

Non-Haar MRA on local fields of positive characteristic

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Abstract

We propose a simple method to construct integral periodic mask and corresponding scaling step functions that generate non-Haar orthogonal MRA on the local field $F^{(s)}$ of positive characteristic p . To construct this mask we use two new ideas. First, we consider local field as vector space over the finite field $GF(p^s)$. Second, we construct scaling function by arbitrary tree that has p^s vertices. By fixed prime number p there exist $p^{s(p^s-2)}$ such trees.

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1 Introduction

First results on the wavelet analysis on local fields received Chinese mathematicians Huikun Jiang, Dengfeng Li, and Ning Jin in the article [1]. They introduced the notion of MRA on local fields, for the fields $F^{(s)}$ of positive characteristic p proved some simple properties and gave an algorithm for constructing wavelets for a known scaling function. Using these results they constructed MRA and corresponding wavelets for the case when a scaling function is the characteristic function of unit ball \mathcal{D} . Such MRA is called usually "Haar MRA" and corresponding wavelets – "Haar wavelets". In [2] wavelet frame on local field are constructed, a necessary condition and sufficient conditions for wavelet frame on local fields are given too. Biswaranjan Behera and Qaiser Jahan [3] constructed the wavelet packets associated with MRA on local fields of positive characteristic. In the article [4] the same authors proved that a function $\varphi \in L^2(\mathbb{F}^{(s)})$ is a scaling function for MRA in $L^2(F^{(s)})$ if and only if

$$\sum_{k \in \mathbb{N}_0} |\hat{\varphi}(\xi + u(k))|^2 = 1 \text{ for a.e. } \xi \in \mathcal{D}, \quad (1.1)$$

$$\lim_{j \rightarrow \infty} |\hat{\varphi}(\mathfrak{p}^j \xi)| = 1 \text{ for a.e. } \xi \in F^{(s)}, \quad (1.2)$$

and there exists an integral periodic function $m_0 \in L^2(\mathcal{D})$ such that

$$\hat{\varphi}(\xi) = m_0(\mathbf{p}\xi)\hat{\varphi}(\mathbf{p}\xi) \text{ for a.e. } \xi \in F^{(s)} \quad (1.3)$$

where $\{u(k)\}$ is the set of shifts, \mathbf{p} is a prime element. B.Behera and Q.Jahan [5] proved also if the translates of the scaling functions of two multiresolution analyses are biorthogonal, then the associated wavelet families are also biorthogonal. So, to construct MRA on a local field $\mathbb{F}^{(s)}$ we must construct an integral periodic mask m_0 with conditions (1.1-1.3). To solve this problem using prime element methods developed in [6] is not simply. Currently there are no effective methods for constructing such masks and scaling functions. In articles [1]-[5] only Haar wavelets are obtained.

In this paper, we propose a simple method to construct integral periodic masks and corresponding scaling step functions that generate non-Haar orthogonal MRA. To construct this mask we use two new ideas. First, we consider local field as vector space over the finite field $GF(p^s)$. Second, we construct a scaling function by arbitrary tree that have p^s nodes. For fixed prime number p there exist $p^{s(p^s-2)}$ such trees.

By $s = 1$ the additive group $F^{(1)+}$ is a Vilenkin group. Issues of constructing of MRA and wavelets on Vilenkin groups may be found in [7]-[12].

The simplest example of a local field of characteristic zero is the field of p -adic numbers. Issues of constructing MRA and wavelets on the field of p -adic numbers can be found in [13]-[15].

The paper is organized as follows. We consider local field $\mathbb{F}^{(s)}$ as a vector space over the finite field $GF(p^s)$. Therefore, in section 2, we recall some concepts and facts from the theory of finite fields and define the local field $\mathbb{F}^{(s)}$ of positive characteristic p as a set of infinite sequences $a = (\mathbf{a}_j)$, where $\mathbf{a}_j \in GF(p^s)$.

In section 3 we prove that local field $\mathbb{F}^{(s)}$ is a vector space over finite field $GF(p^s)$.

In section 4 we prove that the set X of all characters of local field $\mathbb{F}^{(s)}$ also form a vector space over finite field $GF(p^s)$ with product as internal operation and powering as external operation. We define Rademacher functions, find a general view of characters, and prove a basic property of Rademacher functions.

In section 5 we discuss the refinable equation and its mask.

In section 6 we consider refinable equation

$$\hat{\varphi}(\chi) = m_0(\chi)\hat{\varphi}(\chi\mathcal{A}^{-1})$$

with step mask m_0 and find a necessary and sufficient condition under which an integral periodic function m_0 is a mask of some refinement equation.

In section 7 we define (N, M) elementary sets. We prove if $E \subset \mathbb{F}^{(s)}$ is (N, M) elementary set and $|\hat{\varphi}(\chi)| = \mathbf{1}_E(\chi)$ on X then the system of shifts $(\varphi(x \dot{-} h))_{h \in H_0}$ is an orthonormal system.

In section 8 we reduce the problem of construction of step refinable function to construction of some tree. We consider some special class of refinable functions $\varphi(\chi)$ for which $|\hat{\varphi}(\chi)|$ is a characteristic function of a set. We introduce such concepts as "a set generated by a tree" and "a refinable step function generated by a tree" and prove, that every rooted tree containing p^s nodes generates a refinable step function that generate an orthogonal MRA on local field $F^{(s)}$. For $p = s = 2$ we give an example of a refinable step function that generate non-Haar MRA.

Using the results of the article [1] we can construct now corresponding wavelets. This example shows that MRA on local field gives an effective method to construct multidimensional step wavelets.

2 Preliminaries

We will consider two objects: Vilenkin groups and local fields. Let p be a prime number. Vilenkin group $(\mathfrak{G}, \dot{+})$ consists of sequences

$$a = (a_n)_{n \in \mathbb{Z}} = (\dots, a_{n-1}, a_n, a_{n+1}, \dots), \quad a_j = \overline{0, p-1},$$

in which only a finite number of terms with negative numbers are nonzero. The operation $\dot{+}$ is defined as component wise addition modulo p , i.e.

$$a \dot{+} b = (a_n) \dot{+} (b_n) = ((a_n + b_n) \bmod p)_{n \in \mathbb{Z}}.$$

The topology in \mathfrak{G} is determined by subgroups

$$\mathfrak{G}_n = \{a \in \mathfrak{G} : a = (\dots, 0_{n-1}, a_{n-1}, a_n, a_{n+1}, \dots)\}.$$

The equality

$$\rho(a, b) = \begin{cases} \frac{1}{p^n}; & a_m \neq b_n, \quad a_j = b_j \text{ for } j < n \\ 0; & a_j = b_j \text{ for } j \in \mathbb{Z} \end{cases}$$

is the non-Archimedean distance on $(\mathfrak{G}, \dot{+})$. If μ is the Haar measure on \mathfrak{G} then $\mu(\mathfrak{G}_n \dot{+} g) = \mu \mathfrak{G}_n = \frac{1}{p^n}$, $n \in \mathbb{Z}$. The dilation operator \mathcal{A} is defined by the equation

$$\mathcal{A}(a) = (b_n)_{n \in \mathbb{Z}}, \quad b_n = a_{n+1}.$$

It is evident that $\mathcal{A}\mathfrak{G}_n = \mathfrak{G}_{n-1}$ and $\int_{\mathfrak{G}} f(\mathcal{A}u) d\mu = \frac{1}{p} \int_{\mathfrak{G}} f(x) d\mu$.

By a local field we will mean a field K which is locally compact, non-discrete and totally disconnected. We will consider local fields with positive characteristic only. By Pontrjagin–Kovalsky theorem [16] such field is isomorphic to the set $K_L(z)$ of formally Loran series

$$\sum_{n=N}^{\infty} a_n z^n \quad (2.1)$$

with $\mathbf{a}_n \in GF(p^s)$ where $s \in \mathbb{N}$ and p is a prime number. Local field of positive characteristic is denote $F^{(s)}$.

Let $GF(p)$ be a ring (field) of residue class on modulo p . The finite field $GF(p^s)$ consist of vectors $\mathbf{a} = (a^{(0)}, a^{(1)}, \dots, a^{(s-1)})$, where $a^{(j)} \in GF(p)$. The addition operation $(\mathbf{a}) \dot{+} (\mathbf{b})$ is defined coordinate-wise i.e.

$$\mathbf{a} \dot{+} \mathbf{b} = (a^{(j)} + b^{(j)} \bmod p)_{j=0}^{s-1}.$$

To define a product $\mathbf{a}\mathbf{b}$ it is necessary to represent vectors \mathbf{a} and \mathbf{b} as polynomials

$$\mathbf{a} = \sum_{j=0}^{s-1} a^{(j)} t^j, \quad \mathbf{b} = \sum_{j=0}^{s-1} b^{(j)} t^j$$

and multiply these polynomials over the field $GF(p)$. We obtain the polynomial

$$Q(t) = \sum_{j=0}^{s-1} \sum_{k=0}^{s-1} a^{(j)} b^{(k)} t^{j+k} = \sum_{l=0}^{2s-2} t^l \sum_{k,j: k+j=l} a^{(j)} b^{(k)}$$

in which coefficients $\beta_l = \sum_{k,j: k+j=l} a^{(j)} b^{(k)}$ are calculating in the field $GF(p)$.

Then we take a prime polynomial $p_s(t)$ of degree s and divide polynomial $Q(t)$ by $p_s(t)$ over the field $GF(p)$

$$Q(t) = p_s(t)q(t) + H(t).$$

Coefficients b_0, b_1, \dots, b_{s-1} of this rest $H(t)$ are components of product $\mathbf{a}\mathbf{b}$. It is know that a prime polynomial $p_s(t)$ over the field $GF(p)$ exists but not only one. A prime polynomial $p_s(t)$ can be found by exhaustion.

We return to local fields. The sum and product of Loran series (2.1) are defined in the standard way, i.e. if

$$a = \sum_{j=k}^{\infty} \mathbf{a}_j t^j, \quad b = \sum_{j=k}^{\infty} \mathbf{b}_j t^j$$

then

$$a \dot{+} b = \sum_{j=k}^{\infty} (\mathbf{a}_j \dot{+} \mathbf{b}_j) t^j, \quad \mathbf{a}_j \dot{+} \mathbf{b}_j = ((a_j^{(l)} + b_j^{(l)}) \bmod p)_{l=0}^{s-1} \quad (2.2)$$

$$ab = \sum_{l=2k}^{\infty} t^l \sum_{j, \nu: j+\nu=l} \mathbf{a}_j \mathbf{b}_{\nu}. \quad (2.3)$$

Topology in $F^{(s)}$ is given by neighborhood basis of zero

$$F_n^{(s)} = \left\{ a = \sum_{j=n}^{\infty} \mathbf{a}_j t^j : \mathbf{a}_j \in GF(p^s) \right\}.$$

If

$$a = \sum_{j=n}^{\infty} \mathbf{a}_j t^j, \quad \mathbf{a}_n \neq 0$$

then we put by definition $\|a\| = \frac{1}{p^{sn}}$. Consequently

$$F_n^{(s)} = \left\{ x \in F^{(s)} : \|x\| \leq \frac{1}{p^{sn}} \right\}.$$

By $F^{(s)+}$ denote the additive group of field $F^{(s)}$. Neighborhoods $F_n^{(s)}$ are compact subgroups of group $F^{(s)+}$. We will denote them as $F_n^{(s)+}$. The next properties are fulfill

- 1) $\dots \subset F_1^{(s)+} \subset F_0^{(s)+} \subset F_{-1}^{(s)+} \subset \dots$
- 2) $F_n^{(s)+} / F_{n+1}^{(s)+} \cong GF(p^s)$ and $\sharp(F_n^{(s)+} / F_{n+1}^{(s)+}) = p^s$.

Therefore we will assume that a local field $F^{(s)}$ of positive characteristic consists of infinite sequences

$$a = (\dots, \mathbf{0}_{n-1}, \mathbf{a}_n, \mathbf{a}_{n+1}, \dots), \quad \mathbf{a}_j = (a_j^{(0)}, a_j^{(1)}, \dots, a_j^{(s-1)}) \in GF(p^s)$$

in which only finite number of element \mathbf{a}_j with negative numbers are nonzero. The sum and product are defined as

$$a \dot{+} b = (\mathbf{a}_j \dot{+} \mathbf{b}_j)_{j \in \mathbb{Z}}, \quad \mathbf{a}_j \dot{+} \mathbf{b}_j = (a_j^{(\nu)} + b_j^{(\nu)} \bmod p)_{\nu=0}^{s-1}, \quad (2.4)$$

$$ab = \left(\sum_{i,j:i+j=l} (\mathbf{a}_i \mathbf{b}_j) \right)_{l \in \mathbb{Z}} \quad (2.5)$$

In this case

$$\|a\| = \|(\dots, \mathbf{0}_{n-1}, \mathbf{a}_n, \mathbf{a}_{n+1}, \dots)\| = \frac{1}{p^{sn}} \text{ if } \mathbf{a}_n \neq \mathbf{0},$$

$$F_n^{(s)} = \{a = (\mathbf{a}_j)_{j \in \mathbb{Z}} : \mathbf{a}_j \in GF(p^s); \mathbf{a}_j = \mathbf{0} \ \forall j < n\},$$

$$\dots \subset F_1^{(s)} \subset F_0^{(s)} \subset F_{-1}^{(s)} \subset \dots,$$

$F_n^{(s)}$ – are compact subgroups in $F^{(s)+}$ and $\sharp(F_n^{(s)} / F_{n+1}^{(s)}) = p^s$.

It follows that $F^{(1)+}$ is a Vilenkin group. The converse is true also: in Vilenkin group $(\mathfrak{G}, \dot{+})$ we can define product by (2.5). With such operation $(\mathfrak{G}, \dot{+}, \cdot)$ will be a field. Since $F^{(1)+}$ is a Vilenkin group, it follows that

$$1) \int_{\mathfrak{G}_0^\perp} (\chi, x) d\nu(\chi) = \mathbf{1}_{\mathfrak{G}_0}(x), \quad 2) \int_{\mathfrak{G}_0} (\chi, x) d\mu(x) = \mathbf{1}_{\mathfrak{G}_0^\perp}(\chi).$$

$$3) \int_{\mathfrak{G}_n^\perp} (\chi, x) d\nu(\chi) = p^n \mathbf{1}_{\mathfrak{G}_n}(x), \quad 4) \int_{\mathfrak{G}_n} (\chi, x) d\mu(x) = \frac{1}{p^n} \mathbf{1}_{\mathfrak{G}_n^\perp}(\chi)$$

where $\mathfrak{G}_n = F_n^{(1)+}$.

From the definition of $F^{(s)}$ it follows that additive group $F^{(s)+}$ is also Vilenkin group \mathfrak{G} and $F_n^{(s)+} = \mathfrak{G}_{ns}$.

3 Locally field of positive characteristic as vector space over a finite field

Let $(\mathfrak{G}, \dot{+})$ be a Vilenkin group. We can define the multiplication operation on a number $\lambda \in GF(p)$ by the equation

$$a\lambda = \underbrace{a \dot{+} a \dot{+} \dots \dot{+} a}_\lambda.$$

Define the modulus of λ as

$$|\lambda| = \begin{cases} 1, & \lambda \neq 0, \\ 0, & \lambda = 0, \end{cases}$$

and the norm of $a \in \mathfrak{G}$ by the equation

$$\|a\| = p^{-n} \quad (3.1)$$

if

$$a = (\dots 0_{n-1} a_n a_{n+1} \dots), n \in \mathbb{Z}, a_j \in Z_p, a_n \neq 0.$$

Since $GF(p)$ is a field, it follows that $(\mathfrak{G}, \dot{+})$ is a vector space over the field $GF(p)$ and the equation (3.1) defines a norm in $(G, \dot{+}, \cdot \lambda)$.

Now we consider local field $F^{(s)}$ with positive characteristic p . Its elements are infinite sequences

$$a = (\dots, \mathbf{0}_{n-1}, \mathbf{a}_n, \mathbf{a}_{n+1}, \dots), \mathbf{a}_j \in GF(p^s)$$

where

$$\mathbf{a}_j = (a_j^{(0)}, a_j^{(1)}, \dots, a_j^{(s-1)}), a_j^{(\nu)} \in Z_p.$$

Let $\lambda \in GF(p^s)$. By the definition $\|a\| = \frac{1}{p^{sn}}$ if $\mathbf{a}_n \neq \mathbf{0}$. Since

$$\begin{aligned} \lambda a &= (\dots \mathbf{0}_{-1}, \lambda, \mathbf{0}_1, \dots) \cdot (\dots \mathbf{0}_{n-1}, \mathbf{a}_n, \mathbf{a}_{n+1}, \dots) = \\ &= (\lambda + \mathbf{0}x + \mathbf{0}x^2 + \dots)(\mathbf{a}_n x^n + \mathbf{a}_{n+1} x^{n+1} + \dots) = \lambda \mathbf{a}_n x^n + \lambda \mathbf{a}_{n+1} x^{n+1} + \dots = \\ &= (\dots \mathbf{0}_{n-1}, \lambda \mathbf{a}_n, \lambda \mathbf{a}_{n+1}, \dots) \end{aligned}$$

it follows that the product $\lambda \mathbf{a}$ is defined coordinate wise. With such operations $F^{(s)}$ is a vector space. If we define the modulus $|\lambda|$ by the equation

$$|\lambda| = \begin{cases} 1, & \lambda \neq 0, \\ 0, & \lambda = 0. \end{cases}$$

and norm $\|a\|$ by the equation

$$\|a\| = \frac{1}{p^{sn}}, \mathbf{a}_n \neq 0 \quad (3.2)$$

then we can consider the field $F^{(s)}$ as a vector normalized space under the field $GF(p^s)$.

For brevity we denote $K := F^{(s)}$, $K_n := F_n^{(s)}$. Take an element $g \in K_1 \setminus K_2$ and fixed it. It is known [6] that any element $a \in K$ may be written in the form

$$a = \sum_{n \in \mathbb{Z}} \lambda_n g^n, \lambda_n \in U,$$

where U is a fixed full set of coset representatives K_2 in K_1 . We can prove a more general statement.

Theorem 3.1 *Let $(g_n)_{n \in \mathbb{Z}}$ be a fixed basic sequence in K , i.e. $g_n \in K_n \setminus K_{n+1}$. Any element $a \in K$ may be written as sum of the series*

$$a = \sum_{n \in \mathbb{Z}} \bar{\lambda}_n g_n, \bar{\lambda}_n \in GF(p^s). \quad (3.3)$$

Proof. Let $a \in K$. If $a = 0$ then the equation (3.3) is evident. Let $a \neq 0$. Then exists $n \in \mathbb{Z}$ such that $a \in K_n^+ \setminus K_{n+1}^+$. It means that

$$a = (\dots \mathbf{0}_{n-1}, \mathbf{a}_n, \mathbf{a}_{n+1}, \dots), \quad \mathbf{a}_j \in GF(p^s), \mathbf{a}_n \neq \mathbf{0}.$$

Show that there exists $\bar{\lambda}_n \in GF(p^s)$ such that

$$a = \bar{\lambda}_n g_n \dot{+} \alpha_{n+1}, \quad \alpha_{n+1} \in K_{n+1}.$$

Indeed, since $g_n \in K_n \setminus K_{n+1}$ it follows that

$$g_n = (\dots \mathbf{0}_{n-1}, \mathbf{g}_n^{(n)}, \mathbf{g}_{n+1}^{(n)}, \dots), \quad \mathbf{g}_n^{(n)} \neq \mathbf{0}.$$

Since $GF(p^s)$ is a field, it follows there exists $\bar{\lambda}_n \in GF(p^s)$ such that $\bar{\lambda}_n \mathbf{g}_n^{(n)} = \mathbf{a}_n$. Therefore

$$\bar{\lambda}_n g_n = (\dots \mathbf{0}_{n-1}, \bar{\lambda}_n \mathbf{g}_n^{(n)}, \bar{\lambda}_n \mathbf{g}_{n+1}^{(n)}, \dots) = (\dots \mathbf{0}_{n-1}, \mathbf{a}_n, \tilde{\mathbf{a}}_{n+1}, \dots).$$

Consequently

$$a \dot{-} \bar{\lambda}_n g_n = (\dots \mathbf{0}_{n-1}, \mathbf{0}_n, \mathbf{a}_{n+1} - \tilde{\mathbf{a}}_{n+1}, \dots) = \alpha_{n+1} \in K_{n+1}^+,$$

i.e. $a = \bar{\lambda}_n g_n \dot{+} \alpha_{n+1}$. Continuing this process, we obtain (3.3). \square

Corollary. If $g \in K_1 \setminus K_2$ then $g^n \in K_n \setminus K_{n+1}$. Therefore we can take $g_n = g^n$ in the equation (3.3).

Definition 3.1 *The operator*

$$\mathcal{A} : a = \sum_{n \in \mathbb{Z}} \bar{\lambda}_n g_n \mapsto \sum_{n \in \mathbb{Z}} \bar{\lambda}_n g_{n-1}$$

is called a delation operator.

Remark 1. If $g_n = g^n$ and $a = \sum_{n \in \mathbb{Z}} \bar{\lambda}_n g^n$ then $ag^{-1} = \sum_{n \in \mathbb{Z}} \bar{\lambda}_n g^{n-1}$. So the dilation operation may be defined by equation $\mathcal{A}x = g^{-1}x$.

Remark 2. Since additive group $F^{(s)+}$ is Vilenkin group \mathfrak{G} with $F_n^{(s)+} = \mathfrak{G}_{ns}$ it follows that $\mathcal{A}K_n = \mathcal{A}K_{n-1}$ and $\int_{K^+} f(\mathcal{A}u) d\mu = \frac{1}{p^s} \int_{K^+} f(x) d\mu$.

4 Set of characters as vector space over a finite field

Since $F^{(s)+}$ is a Vilenkin group it follows that the set of characters is a locally compact zero-dimensional group with product as group operation

$$(\chi\varphi)(a) = \chi(a) \cdot \varphi(a).$$

Denote the set of characters as X . We want to find the explicit form of characters. Let us define the character r_n in the following way. If

$$a = (\dots, \mathbf{0}_{k-1}, \mathbf{a}_k, \mathbf{a}_{k+1}, \dots), \mathbf{a}_j \in GF(p^s)$$

and

$$\mathbf{a}_j = (a_j^{(0)}, a_j^{(1)}, \dots, a_j^{(s-1)}), a_j^{(\nu)} \in GF(p)$$

then $r_n(a) = e^{\frac{2\pi i}{p} a_k^{(l)}}$, where $n = ks + l$, $0 \leq l < s$.

Lemma 4.1 *Any character $\chi \in X$ can be expressed uniquely as product*

$$\chi = \prod_{n=-\infty}^{+\infty} r_n^{\alpha_n} \quad (\alpha_n = \overline{0, p-1}), \quad (4.1)$$

in which the number of factors with positive numbers are finite.

Proof. Let

$$x = (\dots, \mathbf{0}, \mathbf{x}_j, \dots, \mathbf{x}_k, \mathbf{x}_{k+1}, \dots), \quad \mathbf{x}_k = (x_{ks+0}, x_{ks+1}, \dots, x_{ks+(s-1)})$$

Since $F^{(s)+}$ is a Vilenkin group, it follows that functions $r_{ks+l}(x) = e^{\frac{2\pi i}{p} x_{ks+l}}$, are Rademacher functions on $F^{(s)+}$. Therefore any character χ may be expressed in the form (4.1). \square

Definition 4.1 *Write the character χ as*

$$\chi = \prod_{k \in \mathbb{Z}} r_{ks+0}^{a_k^{(0)}} r_{ks+1}^{a_k^{(1)}} \dots r_{ks+s-1}^{a_k^{(s-1)}}$$

and denote

$$\mathbf{r}_k^{\mathbf{a}_k} := r_{ks+0}^{a_k^{(0)}} r_{ks+1}^{a_k^{(1)}} \dots r_{ks+s-1}^{a_k^{(s-1)}}$$

where $\mathbf{a}_k = (a_k^{(0)}, a_k^{(1)}, \dots, a_k^{(s-1)}) \in GF(p^s)$. The function $\mathbf{r}_k = \mathbf{r}_k^{(1,0,\dots,0)}$ is called Rademacher function.

Definition 4.2 *Assume by the definition*

$$(\mathbf{r}_k^{\mathbf{a}_k})^{\mathbf{b}_k} := \mathbf{r}_k^{\mathbf{a}_k \mathbf{b}_k}, \quad \mathbf{a}_k, \mathbf{b}_k \in GF(p^s).$$

In this case

$$\mathbf{r}_k^{\mathbf{a}_k} = (\mathbf{r}_k^{(1,0,\dots,0)})^{\mathbf{a}_k} = \mathbf{r}_k^{(a_k^{(0)}, a_k^{(1)}, \dots, a_k^{(s-1)})} = r_{ks+0}^{a_k^{(0)}} r_{ks+1}^{a_k^{(1)}} \dots r_{ks+s-1}^{a_k^{(s-1)}}.$$

Therefore we can write χ as the product

$$\chi = \prod_{k \in \mathbb{Z}} \mathbf{r}_k^{\mathbf{a}_k}. \quad (4.2)$$

Definition 4.3 Define $\chi^{\mathbf{b}}$, $\mathbf{b} \in GF(p^s)$ as

$$\chi^{\mathbf{b}} := \prod_{k \in \mathbb{Z}} (\mathbf{r}_k^{\mathbf{a}_k})^{\mathbf{b}_k}.$$

Lemma 4.2 Let \mathbf{r}_k be a Rademacher function. Then

$$\mathbf{r}_k^{\mathbf{u}+\mathbf{v}} = \mathbf{r}_k^{\mathbf{u}} \cdot \mathbf{r}_k^{\mathbf{v}}, \quad \mathbf{u}, \mathbf{v} \in GF(p^s).$$

Proof. Using the definition of Rademacher functions we have for $x = (x_k^{(l)})$

$$\begin{aligned} (\mathbf{r}_k^{\mathbf{u}} \mathbf{r}_k^{\mathbf{v}}, x) &= (\mathbf{r}_k^{\mathbf{u}}, x) (\mathbf{r}_k^{\mathbf{v}}, x) = \prod_{l=0}^{s-1} e^{\frac{2\pi i}{p} u_{ks+l}^{(l)} x_k^{(l)}} \cdot \prod_{l=0}^{s-1} e^{\frac{2\pi i}{p} v_{ks+l}^{(l)} x_k^{(l)}} = \\ &= \prod_{l=0}^{s-1} e^{\frac{2\pi i}{p} (u_{ks+l}^{(l)} + v_{ks+l}^{(l)}) x_k^{(l)}} = (\mathbf{r}_k^{\mathbf{u}+\mathbf{v}}, x). \quad \square \end{aligned}$$

Theorem 4.1 The set of characters of the field $F^{(s)}$ is a vector space $(X, *, \cdot^{GF(p^s)})$ under the finite field $GF(p^s)$ with product as interior operation and powering as exterior operation.

Proof. 1) Check $\chi^{\mathbf{u}+\mathbf{v}} = \chi^{\mathbf{u}} \chi^{\mathbf{v}}$ for $\mathbf{u}, \mathbf{v} \in GF(p^s)$. Let

$$\chi^{\mathbf{u}} = \prod_{k \in \mathbb{Z}} \mathbf{r}_k^{\mathbf{a}_k \mathbf{u}}, \quad \chi^{\mathbf{v}} = \prod_{k \in \mathbb{Z}} \mathbf{r}_k^{\mathbf{a}_k \mathbf{v}}.$$

Using lemma 4.2. we obtain

$$\chi^{\mathbf{u}} \chi^{\mathbf{v}} = \prod_{k \in \mathbb{Z}} \mathbf{r}_k^{\mathbf{a}_k \mathbf{u}} \mathbf{r}_k^{\mathbf{a}_k \mathbf{v}} = \prod_{k \in \mathbb{Z}} \mathbf{r}_k^{\mathbf{a}_k (\mathbf{u}+\mathbf{v})} = \chi^{\mathbf{u}+\mathbf{v}}.$$

2) Check the equation $\chi_1^{\mathbf{u}} \chi_2^{\mathbf{u}} = (\chi_1 \chi_2)^{\mathbf{u}}$. Let

$$\chi_1^{\mathbf{u}} = \prod_{k \in \mathbb{Z}} \mathbf{r}_k^{\mathbf{a}_k \mathbf{u}}, \quad \chi_2^{\mathbf{u}} = \prod_{k \in \mathbb{Z}} \mathbf{r}_k^{\mathbf{b}_k \mathbf{u}}.$$

Using lemma 4.2 we have

$$\chi_1^{\mathbf{u}} \chi_2^{\mathbf{u}} = \prod_{k \in \mathbb{Z}} \mathbf{r}_k^{\mathbf{a}_k \mathbf{u}} \prod_{k \in \mathbb{Z}} \mathbf{r}_k^{\mathbf{b}_k \mathbf{u}} = \prod_{k \in \mathbb{Z}} \mathbf{r}_k^{(\mathbf{a}_k + \mathbf{b}_k) \mathbf{u}} = (\chi_1 \chi_2)^{\mathbf{u}}.$$

3) Since the vector $\mathbf{1} = (1, 0, \dots, 0)$ is a unity element of multiplicative group of the field $GF(p^s)$ it follows that $\chi^{\mathbf{1}} = \chi^{(1,0,\dots,0)} = \prod_{k \in \mathbb{Z}} \mathbf{r}_k^{\mathbf{a}_k \cdot \mathbf{1}} =$

$$\prod_{k \in \mathbb{Z}} \mathbf{r}_k^{\mathbf{a}_k} = \chi.$$

4) The equality $(\chi^{\mathbf{u}})^{\mathbf{v}} = \chi^{\mathbf{uv}}$ is true by the definition.

So, all axioms for exterior operation are fulfil. By lemma 4.2 all axioms for interior operation are fulfil too. \square

It follow from (4.2) that annihilator $(F_k^{(s)})^\perp$ consist from characters of form $\chi = \mathbf{r}_{k-1}^{\mathbf{a}_{k-1}} \mathbf{r}_{k-2}^{\mathbf{a}_{k-2}} \dots$. It is evident also that

- 1) Rademacher system (\mathbf{r}_k) forms a basis of $(X, *, \cdot^{GF(p^s)})$,
- 2) any sequences of characters $\chi_k \in (F_{k+1}^{(s)})^\perp \setminus (F_k^{(s)})^\perp$ forms a basis of $(X, *, \cdot^{GF(p^s)})$.

$$3) (F_k^{(s)})^\perp = \bigsqcup_{\mathbf{a}_{k-1} \in GF(p^s)} (F_{k-1}^{(s)})^\perp \mathbf{r}_{k-1}^{\mathbf{a}_{k-1}}.$$

The next lemma is the basic property of Rademacher functions on local field with positive characteristic.

Lemma 4.3 *Let $g_j = (\dots, \mathbf{0}_{j-1}, (1, 0, \dots, 0)_j, \mathbf{0}_{j+1}, \dots) \in F^{(s)}$, $\mathbf{a}_k, \mathbf{u} \in GF(p^s)$. Then $(\mathbf{r}_k^{\mathbf{a}_k}, \mathbf{u}g_j) = 1$ for any $k \neq j$.*

Proof. Since $\mathbf{u}g_j = (\dots, \mathbf{0}_{j-1}, (u^{(0)}, u^{(1)}, \dots, u^{(s-1)})_j, \mathbf{0}_{j+1}, \dots)$, it follows that

$$(\mathbf{r}_k^{\mathbf{a}_k}, \mathbf{u}g_j) = \prod_{l=0}^{s-1} e^{\frac{2\pi i}{p} a_k^{(l)} u^{(l)}} = \prod_{l=0}^{s-1} e^0 = 1. \quad \square$$

Definition 4.4 *Define a dilation operator \mathcal{A} on the set of characters by the equation $(\chi\mathcal{A}, x) = (\chi, \mathcal{A}x)$.*

Remark. Since additive group $F^{(s)+}$ is Vilenkin group, it follows that $g_j\mathcal{A} = g_{j+1}$, $(K_n^+)^\perp \mathcal{A} = (K_{n+1}^+)^\perp$ and $\int_X f(\chi\mathcal{A}) d\nu = \frac{1}{p^s} \int_X f(\chi) d\nu$.

5 MRA on local fields of positive characteristic

We will use Rademacher function to construct MRA on local fields of positive characteristic. We will assume

$$g_n = (\dots, \mathbf{0}_{n-1}, (1, 0, \dots, 0)_n, \mathbf{0}_{n+1}, \dots).$$

Lemma 5.1 *Let $K = F^{(s)}$ be a local field with characteristic p . Then for any $n \in \mathbb{Z}$*

- 1) $\int_{(K_n^+)^{\perp}} (\chi, x) d\nu(\chi) = p^{sn} \mathbf{1}_{K_n^+}(x),$
 2) $\int_{K_n^+} (\chi, x) d\mu(x) = \frac{1}{p^{sn}} \mathbf{1}_{(K_n^+)^{\perp}}(\chi).$

Proof. First we prove the equation 1). Since K^+ is a zero-dimensional group, it follows

$$\int_{(K_0^+)^{\perp}} (\chi, x) d\nu(\chi) = \mathbf{1}_{K_0^+}(x), \quad \int_{K_0^+} (\chi, x) d\mu(x) = \mathbf{1}_{(K_0^+)^{\perp}}(\chi).$$

By the definition of dilation operator

$$\int_X f(\chi \mathcal{A}) d\nu(\chi) = p^s \int_X f(\chi) d\nu(\chi), \quad \mathbf{1}_{K_n^+}(x) = \mathbf{1}_{K_0^+}(\mathcal{A}^n x).$$

Using these equations we have

$$\begin{aligned} \int_{(K_n^+)^{\perp}} (\chi, x) d\nu(\chi) &= \int_X \mathbf{1}_{(K_n^+)^{\perp}}(\chi) (\chi, x) d\nu(\chi) = \\ &= p^{sn} \int_X (\chi \mathcal{A}^n, x) \mathbf{1}_{(K_n^+)^{\perp}}(\chi \mathcal{A}^n) d\nu(\chi) = \\ &= p^{sn} \int_X (\chi, \mathcal{A}^n x) \mathbf{1}_{(K_0^+)^{\perp}}(\chi) d\nu(\chi) = p^{sn} \mathbf{1}_{K_0^+}(\mathcal{A}^n x) = p^{sn} \mathbf{1}_{K_n^+}(x). \end{aligned}$$

The second equation is proved by analogy. \square

Lemma 5.2 *Let $\chi_{n,l} = \mathbf{r}_n^{\mathbf{a}_n} \mathbf{r}_{n+1}^{\mathbf{a}_{n+1}} \dots \mathbf{r}_{n+l}^{\mathbf{a}_{n+l}}$ be a character does not belong to $(K_n^+)^{\perp}$. Then*

$$\int_{(K_n^+)^{\perp} \chi_{n,l}} (\chi, x) d\nu(\chi) = p^{ns} (\chi_{n,l}, x) \mathbf{1}_{K_n^+}(x).$$

Proof. Denote $\mathfrak{G}_n := K_n^+$. By analogy with previously we have

$$\begin{aligned} \int_{\mathfrak{G}_n^{\perp} \chi_{n,l}} (\chi, x) d\nu(\chi) &= \int_X \mathbf{1}_{\mathfrak{G}_n^{\perp} \chi_{n,l}}(\chi) (\chi, x) d\nu(\chi) = \int_X \mathbf{1}_{\mathfrak{G}_n^{\perp}}(\chi) (\chi_{n,l} \chi, x) d\nu(\chi) = \\ &= \int_{\mathfrak{G}_n^{\perp}} (\chi_{n,l}, x) (\chi, x) d\nu(\chi) = p^{ns} (\chi_{n,l}, x) \mathbf{1}_{\mathfrak{G}_n}(x). \quad \square \end{aligned}$$

Lemma 5.3 Let $h_{n,l} = \mathbf{a}_{n-1}g_{n-1} \dot{+} \mathbf{a}_{n-2}g_{n-2} \dot{+} \dots \dot{+} \mathbf{a}_{n-l}g_{n-l} \notin K_n^+$. Then

$$\int_{K_n^+ \dot{+} h_{n,l}} (\chi, x) d\mu(x) = \frac{1}{p^{ns}} (\chi, h_{n,l}) \mathbf{1}_{(K_n^+)^{\perp}}(\chi).$$

This lemma is dual to lemma 5.2.

Definition 5.1 Let $M, N \in \mathbb{N}$. Denote by $\mathfrak{D}_M(K_{-N})$ the set of step-functions $f \in L_2(K)$ such that 1) $\text{supp } f \subset K_{-N}$, and 2) f is constant on cosets $K_M \dot{+} g$. Similarly is defined $\mathfrak{D}_{-N}(K_M^{\perp})$.

Lemma 5.4 Let $M, N \in \mathbb{N}$. $f \in \mathfrak{D}_M(K_{-N})$ if and only if $\hat{f} \in \mathfrak{D}_{-N}(K_M^{\perp})$.

Proof. It is evident since additive group F^+ is Vilenkin group.

Lemma 5.5 Let $\varphi \in L_2(K)$. The system $(\varphi(x \dot{-} h))_{h \in H_0}$ is orthonormal if and only if the system $(p^{\frac{ns}{2}} \varphi(\mathcal{A}^n x \dot{-} h))_{h \in H_0}$ is orthonormal.

Proof. This lemma follows from the equation

$$\int_K p^{\frac{ns}{2}} \varphi(\mathcal{A}^n x \dot{-} h) p^{\frac{ns}{2}} \overline{\varphi(\mathcal{A}^n x \dot{-} g)} d\mu = \int_K \varphi(x \dot{-} h) \overline{\varphi(x \dot{-} g)} d\mu. \quad \square$$

Definition 5.2 A family of closed subspaces V_n , $n \in \mathbb{Z}$, is said to be a multiresolution analysis of $L_2(K)$ if the following axioms are satisfied:

A1) $V_n \subset V_{n+1}$;

A2) $\overline{\bigcup_{n \in \mathbb{Z}} V_n} = L_2(K)$ and $\bigcap_{n \in \mathbb{Z}} V_n = \{0\}$;

A3) $f(x) \in V_n \iff f(\mathcal{A}x) \in V_{n+1}$ (\mathcal{A} is a dilation operator);

A4) $f(x) \in V_0 \implies f(x \dot{-} h) \in V_0$ for all $h \in H_0$; (H_0 is analog of \mathbb{Z}).

A5) there exists a function $\varphi \in L_2(K)$ such that the system $(\varphi(x \dot{-} h))_{h \in H_0}$ is an orthonormal basis for V_0 .

A function φ occurring in axiom A5 is called a scaling function.

It is clear that the axiom A5 follows the axiom A4. Next we will follow the conventional approach. Let $\varphi(x) \in L_2(K)$, and suppose that $(\varphi(x \dot{-} h))_{h \in H_0}$ is an orthonormal system in $L_2(K)$. With the function φ and the dilation operator \mathcal{A} , we define the linear subspaces $L_n = (\varphi(\mathcal{A}x \dot{-} h))_{h \in H_0}$ and closed subspaces $V_n = \overline{L_n}$. It is evident that the functions $p^{\frac{ns}{2}} \varphi(\mathcal{A}x \dot{-} h)_{h \in H_0}$ form

an orthonormal basis for V_n , $n \in \mathbb{Z}$. Therefore the axiom A3 is fulfilled. If subspaces V_j form a MRA, then the function φ is said to *generate* an MRA in $L_2(K)$. If a function φ generates an MRA, then we obtain from the axiom A1

$$\varphi(x) = \sum_{h \in H_0} \beta_h \varphi(\mathcal{A}x - h) \quad \left(\sum |\beta_h|^2 < +\infty \right). \quad (5.1)$$

Therefore we will look up a function $\varphi \in L_2(K)$, which generates an MRA in $L_2(K)$, as a solution of the refinement equation (5.1). A solution of refinement equation (5.1) is called a *refinable function*.

Lemma 5.6 *Let $\varphi \in \mathfrak{D}_M(K_{-N})$ be a solution of (5.1). Then*

$$\varphi(x) = \sum_{h \in H_0^{(N+1)}} \beta_h \varphi(\mathcal{A}x - h) \quad (5.2)$$

The proof repeats the proof of Lemma 4.1 in [10].

Theorem 5.1 *Let $\varphi \in \mathfrak{D}_M(K_{-N})$ and let $(\varphi(x - h))_{h \in H_0}$ be an orthonormal system. $V_n \subset V_{n+1}$ if and only if the function $\varphi(x)$ is a solution of refinement equation (5.2).*

The proof repeats the proof of Theorem 4.2 in [10].

Theorem 5.2 ([4], th.4.1). *Let $\varphi \in \mathfrak{D}_M(K_{-N})$ be a solution of the equation (5.2), $(\varphi(x - h))_{h \in H_0}$ an orthonormal basis in V_0 . Then $\bigcap_{n \in \mathbb{Z}} V_n = \{0\}$.*

Theorem 5.3 ([4], th.4.3) *Let $\varphi \in \mathfrak{D}_M(K_{-N})$ be a solution of the equation (5.2), $(\varphi(x - h))_{h \in H_0}$ an orthonormal basis in V_0 , and $\hat{\varphi}(0) \neq 0$. Then $\overline{\bigcup_{n \in \mathbb{Z}} V_n} = L_2(K)$.*

The refinement equation (5.2) may be written in the form

$$\hat{\varphi}(\chi) = m_0(\chi) \hat{\varphi}(\chi \mathcal{A}^{-1}), \quad (5.3)$$

where

$$m_0(\chi) = \frac{1}{p^s} \sum_{h \in H_0^{(N+1)}} \beta_h \overline{\varphi(\chi \mathcal{A}^{-1} - h)} \quad (5.4)$$

is a mask of the equation (5.3).

Lemma 5.7 *Let $\varphi \in \mathfrak{D}_M(K_{-N})$. Then the mask $m_0(\chi)$ is constant on cosets $K_{-N}^\perp \zeta$.*

Proof. We will prove that $(\chi, \mathcal{A}^{-1}h)$ are constant on cosets $K_{-N}^\perp \zeta$. Without loss of generality, we can assume that $\zeta = \mathbf{r}_{-N}^{\mathbf{a}_{-N}} \dots \mathbf{r}_{-N+s}^{\mathbf{a}_{-N+s}} \notin K_{-N}^\perp$. If

$$h = \mathbf{a}_{-1}g_{-1} \dot{+} \dots \dot{+} \mathbf{a}_{-N-1}g_{-N-1} \in H_0^{(N+1)}, \quad \mathbf{a}_j \in GF(p^s)$$

then

$$\mathcal{A}^{-1}h = \mathbf{a}_{-1}g_0 \dot{+} \dots \dot{+} \mathbf{a}_{-N-1}g_{-N} \in K_{-N}.$$

If $\chi \in K_{-N}^\perp \zeta$ then $\chi = \chi_{-N} \zeta$ where $\chi_{-N} \in K_{-N}^\perp$. Therefore $(\chi, \mathcal{A}^{-1}h) = (\chi_{-N} \zeta, \mathcal{A}^{-1}h) = (\zeta, \mathcal{A}^{-1}h)$. This means that $(\chi, \mathcal{A}^{-1}h)$ depends on ζ only. \square

Lemma 5.8 *The mask $m_0(\chi)$ is a periodic function with any period $\mathbf{r}_1^{\mathbf{a}_1} \mathbf{r}_2^{\mathbf{a}_2} \dots \mathbf{r}_l^{\mathbf{a}_l}$ ($l \in \mathbb{N}$, $\mathbf{a}_j \in GF(p^s)$, $j = \overline{1, l}$).*

Proof. Using the equation $(\mathbf{r}_k^{\mathbf{b}_k}, \mathbf{u}g_j) = 1, (k \neq j)$ we find

$$\begin{aligned} (\chi \mathbf{r}_1^{\mathbf{b}_1} \mathbf{r}_2^{\mathbf{b}_2} \dots \mathbf{r}_l^{\mathbf{b}_l}, \mathcal{A}^{-1}h) &= (\chi \mathbf{r}_1^{\mathbf{b}_1} \mathbf{r}_2^{\mathbf{b}_2} \dots \mathbf{r}_l^{\mathbf{b}_l}, \mathbf{a}_{-1}g_0 \dot{+} \mathbf{a}_{-2}g_{-1} \dot{+} \dots \dot{+} \mathbf{a}_{-N-1}g_{-N}) = \\ &= (\chi, \mathbf{a}_{-1}g_0 \dot{+} \mathbf{a}_{-2}g_{-1} \dot{+} \dots \dot{+} \mathbf{a}_{-N-1}g_{-N}) = (\chi \mathcal{A}^{-1}, h). \end{aligned}$$

Therefore $m_0(\chi \mathbf{r}_1^{\mathbf{b}_1} \mathbf{r}_2^{\mathbf{b}_2} \dots \mathbf{r}_l^{\mathbf{b}_l}) = m_0(\chi)$ and the lemma is proved. \square

Lemma 5.9 *The mask $m_0(\chi)$ is defined by its values on cosets $K_{-N}^\perp \mathbf{r}_{-N}^{\mathbf{a}_{-N}} \dots \mathbf{r}_0^{\mathbf{a}_0}$ ($\mathbf{a}_j = (a_j^{(0)}, a_j^{(1)}, \dots, a_j^{(s-1)}) \in GF(p^s)$).*

Proof. Let us denote

$$\begin{aligned} k &= \sum_{j=0}^N (a_{-j}^{(0)} + a_{-j}^{(1)}p + \dots + a_{-j}^{(s-1)}p^{s-1})p^{sj} \in [0, p^{s(N+1)} - 1], \\ l &= \sum_{j=1}^{N+1} (\alpha_{-j}^{(0)} + \alpha_{-j}^{(1)}p + \dots + \alpha_{-j}^{(s-1)}p^{s-1})p^{s(j-1)} \in [0, p^{s(N+1)} - 1]. \end{aligned}$$

Then (5.4) can be written as the system

$$m_0(\chi_k) = \frac{1}{p^s} \sum_{l=0}^{p^{s(N+1)}-1} \beta_l(\overline{\chi_k, \mathcal{A}^{-1}h_l}), \quad k = \overline{0, p^{s(N+1)}-1} \quad (5.5)$$

in the unknowns β_l . We consider the characters χ_k on the subgroup K_{-N}^+ . Since $\mathcal{A}^{-1}h_l \in K_{-N}^+$, it follows that the matrix $p^{-\frac{s(N+1)}{2}}(\chi_k, \overline{\mathcal{A}^{-1}h_l})$ is unitary, and so the system (5.5) has a unique solution for each finite sequence $(m_0(\chi_k))_{k=0}^{p^{s(N+1)}-1}$. \square

Remark. The function $m_0(\chi)$ constructing in Lemma 5.9 may be not a mask for $\varphi \in \mathfrak{D}_M(\mathfrak{G}_{-N})$. In the section 4 we find conditions under which the function $m_0(\chi)$ will be a mask.

Lemma 5.10 *Let $\hat{f}_0(\chi) \in \mathfrak{D}_{-N}((K_1^+)^{\perp})$. Then*

$$\hat{f}_0(\chi) = \frac{1}{p^s} \sum_{h \in H_0^{(N+1)}} \beta_h \overline{(\chi, \mathcal{A}^{-1}h)}. \quad (5.6)$$

Prof. Since $\int_{(K_0^+)^{\perp}} (\chi, g) \overline{(\chi, h)} d\nu(\chi) = \delta_{h,g}$ for $h, g \in H_0$ it follows that

$\int_{(K_1^+)^{\perp}} (\chi \mathcal{A}^{-1}, g) \overline{(\chi \mathcal{A}^{-1}, h)} d\nu(\chi) = p \delta_{h,g}$. Therefore we can consider the set $\left(\frac{\mathcal{A}^{-1}h}{\sqrt{p^s}}\right)_{h \in H_0^{(N+1)}}$ as an orthonormal system on $(K_1^+)^{\perp}$. We know (lemma 5.7) that $(\chi, \mathcal{A}^{-1}h)$ is a constant on cosets $(K_{-N}^+)^{\perp}\zeta$. It is evident the dimensional of $\mathfrak{D}_{-N}((K_1^+)^{\perp})$ is equal to $p^{s(N+1)}$. Therefore the system $\left(\frac{\mathcal{A}^{-1}h}{\sqrt{p^s}}\right)_{h \in H_0^{(N+1)}}$ is an orthonormal basis for $\mathfrak{D}_{-N}((K_1^+)^{\perp})$ and the equation (5.6) is valid. \square

6 Non Haar wavelets

In this section we find the necessary and sufficient condition under which a step function $\varphi(x) \in \mathfrak{D}_M(K_{-N})$ generates an orthogonal MRA on the local field with positive characteristic. We will prove also that for any $n \in \mathbb{N}$ there exists a step function φ such that 1) φ generate an orthogonal MRA, 2) $\text{supp } \hat{\varphi} \subset K_n^{\perp}$, 3) $\hat{\varphi}(K_n^{\perp} \setminus K_{n-1}^{\perp}) \not\equiv 0$.

Note that the results of Sections 6, 7 and 8, there are analogues of the corresponding results for Vilenkin groups [12]. Moreover, we use the same methods. This is possible since the basic property of Rademacher functions (Lemma 4.3) is satisfied.

First we give a test under which the system of shifts $(\varphi(x-h))_{h \in H_0}$ is an orthonormal system.

Theorem 6.1 Let $\varphi(x) \in \mathfrak{D}_M(K_{-N})$. A shift's system $(\varphi(x \dot{-} h))_{h \in H_0}$ will be orthonormal if and only if for any $\bar{\alpha}_{-N}, \bar{\alpha}_{-N+1}, \dots, \bar{\alpha}_{-1} \in GF(p^s)$

$$\sum_{\bar{\alpha}_0, \bar{\alpha}_1, \dots, \bar{\alpha}_{M-1} \in GF(p^s)} |\hat{\varphi}((K_{-N}^+)^{\perp} \mathbf{r}_{-N}^{\bar{\alpha}_{-N}} \dots \mathbf{r}_0^{\bar{\alpha}_0} \dots \mathbf{r}_{M-1}^{\bar{\alpha}_{M-1}})|^2 = 1. \quad (6.1)$$

Proof. First we prove that the system $(\varphi(x \dot{-} h))_{h \in H_0}$ will be orthonormal if and only if

$$\sum_{\bar{\alpha}_{-N}, \dots, \bar{\alpha}_0, \dots, \bar{\alpha}_{M-1}} |\hat{\varphi}((K_{-N}^+)^{\perp} \mathbf{r}_{-N}^{\bar{\alpha}_{-N}} \dots \mathbf{r}_{M-1}^{\bar{\alpha}_{M-1}})|^2 = p^{Ns}. \quad (6.2)$$

and for any vector $(\mathbf{a}_{-1}, \mathbf{a}_{-2}, \dots, \mathbf{a}_{-N}) \neq (0, 0, \dots, 0)$, $(\mathbf{a}_j \in GF(p^s))$

$$\begin{aligned} \sum_{\bar{\alpha}_{-1}, \dots, \bar{\alpha}_{-N}} \exp \left(\frac{2\pi i}{p} (\mathbf{a}_{-1}^{(0)} \alpha_{-1}^{(0)} + \dots + \mathbf{a}_{-1}^{(s-1)} \alpha_{-1}^{(s-1)} + \dots + \mathbf{a}_{-N}^{(0)} \alpha_{-N}^{(0)} + \dots + \mathbf{a}_{-N}^{(s-1)} \alpha_{-N}^{(s-1)}) \right) \times \\ \times \sum_{\bar{\alpha}_0, \bar{\alpha}_1, \dots, \bar{\alpha}_{M-1}} |\hat{\varphi}((K_{-N}^+)^{\perp} \mathbf{r}_{-N}^{\bar{\alpha}_{-N}} \dots \mathbf{r}_{M-1}^{\bar{\alpha}_{M-1}})|^2 = 0 \end{aligned} \quad (6.3)$$

Let $(\varphi(x \dot{-} h))_{h \in H_0}$ be an orthonormal system. Using the Plancherel equality and Lemma 5.1 we have

$$\begin{aligned} \delta_{h_1 h_2} &= \int_{K^+} \varphi(x \dot{-} h_1) \overline{\varphi(x \dot{-} h_2)} d\mu(x) = \int_{(K_M^+)^{\perp}} |\hat{\varphi}(\chi)|^2 (\chi, h_2 \dot{-} h_1) d\nu(\chi) = \\ &= \sum_{\bar{\alpha}_{-N}, \dots, \bar{\alpha}_0, \dots, \bar{\alpha}_{M-1}} \int_{(K_{-N}^+)^{\perp} \mathbf{r}_{-N}^{\bar{\alpha}_{-N}} \dots \mathbf{r}_0^{\bar{\alpha}_0} \dots \mathbf{r}_{M-1}^{\bar{\alpha}_{M-1}}} |\hat{\varphi}(\chi)|^2 (\chi, h_2 \dot{-} h_1) d\nu(\chi) = \\ &= \sum_{\bar{\alpha}_{-N}, \dots, \bar{\alpha}_{M-1}} |\hat{\varphi}((K_{-N}^+)^{\perp} \mathbf{r}_{-N}^{\bar{\alpha}_{-N}} \dots \mathbf{r}_{M-1}^{\bar{\alpha}_{M-1}})|^2 \int_{(K_{-N}^+)^{\perp} \mathbf{r}_{-N}^{\bar{\alpha}_{-N}} \dots \mathbf{r}_{M-1}^{\bar{\alpha}_{M-1}}} (\chi, h_2 \dot{-} h_1) d\nu(\chi) = \\ &= p^{-sN} \mathbf{1}_{K_{-N}^+}(h_2 \dot{-} h_1) \times \\ &\times \sum_{\bar{\alpha}_{-N}, \dots, \bar{\alpha}_{M-1}} |\hat{\varphi}((K_{-N}^+)^{\perp} \mathbf{r}_{-N}^{\bar{\alpha}_{-N}} \dots \mathbf{r}_0^{\bar{\alpha}_0} \dots \mathbf{r}_{M-1}^{\bar{\alpha}_{M-1}})|^2 (\mathbf{r}_{-N}^{\bar{\alpha}_{-N}} \dots \mathbf{r}_0^{\bar{\alpha}_0} \dots \mathbf{r}_{M-1}^{\bar{\alpha}_{M-1}}, h_2 \dot{-} h_1). \end{aligned}$$

If $h_2 = h_1$, we obtain the equality (6.2). If $h_2 \neq h_1$ then

$$h_2 \dot{-} h_1 = \mathbf{a}_{-1} g_{-1} \dot{+} \dots \dot{+} \mathbf{a}_{-N} g_{-N} \in K_{-N}^+ \quad (6.4)$$

or

$$h_2 \dot{-} h_1 = \mathbf{a}_{-1} g_{-1} \dot{+} \dots \dot{+} \mathbf{a}_{-N} g_{-N} \dot{+} \dots \dot{+} \mathbf{a}_{-l} g_{-l} \in K^+ \setminus K_{-N}^+. \quad (6.5)$$

If the condition (6.5) are fulfilled, then $\mathbf{1}_{(K_{-N}^+)}(h_2 - h_1) = 0$. If the condition (6.4) are fulfilled, then

$$\mathbf{1}_{K_{-N}^+}(h_2 - h_1) = 1,$$

$$(\mathbf{r}_{-N}^{\alpha_{-N}} \dots \mathbf{r}_0^{\bar{\alpha}_0} \dots \mathbf{r}_{M-1}^{\alpha_{M-1}}, h_2 \dots h_1) = (\mathbf{r}_{-N}^{\mathbf{a}_{-N}}, \bar{\alpha}_{-N} g_{-N}) \dots (\mathbf{r}_{-1}^{\mathbf{a}_{-1}}, \bar{\alpha}_{-1} g_{-1}).$$

Using the equality $(\mathbf{r}_k^{\mathbf{a}_k}, \mathbf{u}g_k) = \prod_{l=0}^{s-1} e^{\frac{2\pi i}{p} u^{(l)} a_k^{(l)}}$ we obtain the equality (6). The conversely may be proved by analogy.

Let us show now if for any vector $(\mathbf{a}_{-1}, \mathbf{a}_{-2}, \dots, \mathbf{a}_{-N}) \neq (0, 0, \dots, 0)$ the conditions (6.2) (6) are fulfilled, then for any $\bar{\alpha}_{-N}, \bar{\alpha}_{-N+1}, \dots, \bar{\alpha}_{-1} \in FG(p^s)$

$$\sum_{\bar{\alpha}_0, \bar{\alpha}_1, \dots, \bar{\alpha}_{M-1}} |\hat{\varphi}((K_{-N}^+)^{\perp} \mathbf{r}_{-N}^{\bar{\alpha}_{-N}} \dots \mathbf{r}_0^{\bar{\alpha}_0} \dots \mathbf{r}_{M-1}^{\bar{\alpha}_{M-1}})|^2 = 1. \quad (6.6)$$

Let us denote

$$n = \sum_{j=1}^N \sum_{l=0}^{s-1} a_{-j}^{(l)} p^{s(j-1)}, \quad k = \sum_{j=1}^N \sum_{l=0}^{s-1} \alpha_{-j}^{(l)} p^{s(j-1)},$$

$$C_{n,k} = \exp \frac{2\pi i}{p} \left(\sum_{j=1}^N \sum_{l=0}^{s-1} \alpha_{-j}^{(l)} a_{-j}^{(l)} \right).$$

and write the equalities (6.2) and (6) as the system

[illegible]

with unknowns

$$x_k = \sum_{\bar{\alpha}_0, \bar{\alpha}_1, \dots, \bar{\alpha}_{M-1}} |\hat{\varphi}((K_{-N}^+)^{\perp} \mathbf{r}_{-N}^{\bar{\alpha}_{-N}} \dots \mathbf{r}_0^{\bar{\alpha}_0} \dots \mathbf{r}_{M-1}^{\bar{\alpha}_{M-1}})|^2.$$

The matrix $(C_{n,k})$ is orthogonal. Indeed, if $(\mathbf{a}_{-1}, \mathbf{a}_{-2}, \dots, \mathbf{a}_{-N}) \neq (\mathbf{a}'_{-1}, \mathbf{a}'_{-2}, \dots, \mathbf{a}'_{-N})$, i.e., $n \neq n'$ we obtain

$$\sum_{k=0}^{p^N-1} C_{n,k} \overline{C_{n',k}} = \sum_{\bar{\alpha}_{-1}, \dots, \bar{\alpha}_{-N}} \exp \left(\frac{2\pi i}{p} \sum_{j=1}^N \sum_{l=0}^{s-1} (a_{-j}^{(l)} - a_{-j}'^{(l)}) \alpha_{-j}^{(l)} \right) = 0,$$

so at least one of differences $\mathbf{a}_{-l} - \mathbf{a}'_{-l} \neq 0$. So, the system (6.7) has unique solution. It is evident that $x_k = 1$ is a solution of this system. This means

that (6.6) is fulfilled, and the necessity is proved. The sufficiency is evident. \square

Now we obtain a necessary and sufficient conditions for function $m_0(\chi)$ to be a mask on the class $\mathfrak{D}_{-N}((K_M^+)^{\perp})$, i.e. there exists $\hat{\varphi} \in \mathfrak{D}_{-N}((K_M^+)^{\perp})$ for which

$$\hat{\varphi}(\chi) = m_0(\chi)\hat{\varphi}(\chi\mathcal{A}^{-1}). \quad (6.8)$$

If $m_0(\chi)$ is a mask of (6.8) then

P1) $m_0(\chi)$ is constant on cosets $(K_{-N}^+)^{\perp}\zeta$,

P2) $m_0(\chi)$ is periodic with any period $\mathbf{r}_1^{\bar{\alpha}_1}\mathbf{r}_2^{\bar{\alpha}_2}\dots\mathbf{r}_l^{\bar{\alpha}_l}$, $\bar{\alpha}_j \in GF(p^s)$,

P3) $m_0((K_{-N}^+)^{\perp}) = 1$.

Therefore we will assume that m_0 satisfies these conditions. Let

$$E_k \subset (K_k^+)^{\perp} \setminus (K_{k-1}^+)^{\perp}, \quad (k = -N+1, -N+2, \dots, 0, 1, \dots, M, M+1)$$

be a set, on which $m_0(E_k) = 0$. Since $m_0(\chi)$ is constant on cosets $(K_{-N}^+)^{\perp}\zeta$, it follows that E_k is a union of such cosets or $E_k = \emptyset$.

Theorem 6.2 $m_0(\chi)$ is a mask of some equation on the class $\mathfrak{D}_{-N}((K_M^+)^{\perp})$ if and only if

$$m_0(\chi)m_0(\chi\mathcal{A}^{-1})\dots m_0(\chi\mathcal{A}^{-M-N}) = 0 \quad (6.9)$$

on $(K_{M+1}^+)^{\perp} \setminus (K_M^+)^{\perp}$.

Proof. Indeed, if (6.9) is true we set

$$\hat{\varphi}(\chi) = \prod_{k=0}^{\infty} m_0(\chi\mathcal{A}^{-k}) \in \mathfrak{D}_{-N}((K_M^+)^{\perp}).$$

Then $\hat{\varphi}(\chi) = m_0(\chi)\hat{\varphi}(\chi\mathcal{A}^{-1})$ and

$$m_0(\chi) = \sum_{h \in H_0^{(N+1)}} \beta_h \overline{(\chi\mathcal{A}^{-1}, h)}$$

for some β_h . Therefore $m_0(\chi)$ is a mask. Inversely let $m_0(\chi)$ be a mask, i.e. $\hat{\varphi}(\chi) = m_0(\chi)\hat{\varphi}(\chi\mathcal{A}^{-1}) \in \mathfrak{D}_{-N}((K_M^+)^{\perp})$. From it we find

$$\hat{\varphi}(\chi) = m_0(\chi)m_0(\chi\mathcal{A}^{-1})\dots m_0(\chi\mathcal{A}^{-M-N})\hat{\varphi}(\chi\mathcal{A}^{-M-N-1}),$$

and $\hat{\varphi}(\chi\mathcal{A}^{-M-N-1}) = 1$ on $(K_{M+1}^+)^{\perp}$. Since $\hat{\varphi}(\chi) = 0$ on $(K_{M+1}^+)^{\perp} \setminus (K_M^+)^{\perp}$, it follows

$$m_0(\chi)m_0(\chi\mathcal{A}^{-1})\dots m_0(\chi\mathcal{A}^{-M-N}) = 0$$

on $(K_{M+1}^+)^{\perp} \setminus (K_M^+)^{\perp}$. \square

Lemma 6.1 Let $\hat{\varphi} \in \mathfrak{D}_{-N}((K_M^+)^{\perp})$ be a solution of the refinement equation

$$\hat{\varphi}(\chi) = m_0(\chi)\hat{\varphi}(\chi\mathcal{A}^{-1})$$

and $(\varphi(x\dot{-}h))_{h \in H_0}$ be an orthonormal system. Then for any

$$\bar{\alpha}_{-N}, \alpha_{-N+1}, \dots, \bar{\alpha}_{-1} \in GF(p^s)$$

$$\sum_{\bar{\alpha}_0 \in GF(p^s)} |m_0((K_{-N}^+)^{\perp} \mathbf{r}_{-N}^{\bar{\alpha}_{-N}} \mathbf{r}_{-N+1}^{\bar{\alpha}_{-N+1}} \dots \mathbf{r}_{-1}^{\bar{\alpha}_{-1}} \mathbf{r}_0^{\bar{\alpha}_0})|^2 = 1. \quad (6.10)$$

Proof. Since $\hat{\varphi} \in \mathfrak{D}_{-N}((K_M^+)^{\perp})$, it follows that $\hat{\varphi}((K_{M+1}^+)^{\perp} \setminus ((K_M^+)^{\perp}) = 0$. Using theorem 6.1 we have

$$\begin{aligned} 1 &= \sum_{\bar{\alpha}_0, \bar{\alpha}_1, \dots, \bar{\alpha}_{M-1} \in GF(p^s)} |\hat{\varphi}((K_{-N}^+)^{\perp} \mathbf{r}_{-N}^{\bar{\alpha}_{-N}} \dots \mathbf{r}_0^{\bar{\alpha}_0} \dots \mathbf{r}_{M-1}^{\bar{\alpha}_{M-1}})|^2 = \\ &= \sum_{\bar{\alpha}_0, \dots, \bar{\alpha}_{M-1}, \bar{\alpha}_M \in GF(p^s)} |\hat{\varphi}((K_{-N}^+)^{\perp} \mathbf{r}_{-N}^{\bar{\alpha}_{-N}} \dots \mathbf{r}_0^{\bar{\alpha}_0} \dots \mathbf{r}_{M-1}^{\bar{\alpha}_{M-1}} \mathbf{r}_M^{\bar{\alpha}_M})|^2 = \sum_{\bar{\alpha}_0=0}^{p-1} |m_0((K_{-N}^+)^{\perp} \mathbf{r}_{-N}^{\bar{\alpha}_{-N}} \dots \mathbf{r}_0^{\bar{\alpha}_0})|^2 \\ &\quad \cdot \sum_{\bar{\alpha}_1, \dots, \bar{\alpha}_{M-1}, \bar{\alpha}_M \in GF(p^s)} |\hat{\varphi}((K_{-N}^+)^{\perp} \mathbf{r}_{-N}^{\bar{\alpha}_{-N}+1} \dots \mathbf{r}_{-1}^{\bar{\alpha}_0} \mathbf{r}_0^{\bar{\alpha}_1} \dots \mathbf{r}_{M-2}^{\bar{\alpha}_{M-1}} \mathbf{r}_{M-1}^{\bar{\alpha}_M})|^2 = \\ &= \sum_{\bar{\alpha}_0 \in GF(p^s)} |m_0((K_{-N}^+)^{\perp} \mathbf{r}_{-N}^{\bar{\alpha}_{-N}} \dots \mathbf{r}_0^{\bar{\alpha}_0})|^2. \quad \square \end{aligned}$$

Theorem 6.3 Suppose the function $m_0(\chi)$ satisfies the conditions P1, P2, P3, (6.9), and the function

$$\hat{\varphi}(\chi) = \prod_{n=0}^{\infty} m_0(\chi\mathcal{A}^{-n})$$

satisfies the condition (6.1). Then $\varphi \in \mathfrak{D}_M(K_{-N}^+)$ and generates an orthogonal MRA.

Proof. It is evident that $\hat{\varphi} \in \mathfrak{D}_{-N}((K_M^+)^{\perp})$, $\hat{\varphi}(\chi) = m_0(\chi)\hat{\varphi}(\chi\mathcal{A}^{-1})$, and $(\varphi(x\dot{-}h))_{h \in H_0}$ is an orthonormal system. From theorems 5.1, 5.2, 5.3 we find that the function φ generates an orthogonal MRA. \square

7 (N,M)-elementary sets

So, to find a refinable function that generates orthogonal MRA, we need take a function $m_0(\chi)$ that satisfies conditions P1, P2, P3, (6.9), construct the function

$$\hat{\varphi}(\chi) = \prod_{k=0}^{\infty} m_0(\chi\mathcal{A}^{-k}) \in \mathfrak{D}_{-N}((K_M^+)^{\perp})$$

and check that the system $\varphi(x \dot{-} h)_{h \in H_0}$ is orthonormal. We want to give a simple condition under which the system of shifts $\varphi(x \dot{-} h)_{h \in H_0}$ is orthonormal.

Definition 7.1 *Let $N, M \in \mathbb{N}$. A set $E \subset X$ is called (N, M) -elementary if E is disjoint union of p^{sN} cosets*

$$(K_{-N}^+)^{\perp} \zeta_j = (K_{-N}^+)^{\perp} \underbrace{\mathbf{r}_{-N}^{\bar{\alpha}_{-N}} \mathbf{r}_{-N+1}^{\bar{\alpha}_{-N+1}} \dots \mathbf{r}_{-1}^{\bar{\alpha}_{-1}}}_{\xi_j} \underbrace{\mathbf{r}_0^{\bar{\alpha}_0} \dots \mathbf{r}_{M-1}^{\bar{\alpha}_{M-1}}}_{\eta_j} = (K_{-N}^+)^{\perp} \xi_j \eta_j,$$

$j = 0, 1, \dots, p^{sN} - 1, j = \sum_{l=0}^{N-1} (\alpha_{-N+l}^{(0)} + \alpha_{-N+l}^{(1)} p + \dots + \alpha_{-N+l}^{(s-1)} p^{s-1}) p^{sl}$
 $(\bar{\alpha}_\nu \in GF(p^s))$ such that

- 1) $\bigsqcup_{j=0}^{p^{sN}-1} (K_{-N}^+)^{\perp} \xi_j = (K_0^+)^{\perp}, (K_{-N}^+)^{\perp} \xi_0 = (K_{-N}^+)^{\perp},$
- 2) for any $l = \overline{0, M+N-1}$ the intersection $((K_{-N+l+1}^+)^{\perp} \setminus (K_{-N+l}^+)^{\perp}) \cap E \neq \emptyset.$

Lemma 7.1 *The set $H_0 \subset K$ is an orthonormal system on any (N, M) -elementary set $E \subset X$.*

Proof. Let $h, g \in H_0$. Using the definition of (N, M) -elementary set we have

$$\begin{aligned} \int_E (\chi, h) \overline{(\chi, g)} d\nu(\chi) &= \sum_{j=0}^{p^{sN}-1} \int_{(K_{-N}^+)^{\perp} \zeta_j} (\chi, h) \overline{(\chi, g)} d\nu(\chi) = \\ &= \sum_{j=0}^{p^{sN}-1} \int_X \mathbf{1}_{(K_{-N}^+)^{\perp} \zeta_j}(\chi) (\chi, h) \overline{(\chi, g)} d\nu(\chi) = \\ &= \sum_{j=0}^{p^{sN}-1} \int_X \mathbf{1}_{(K_{-N}^+)^{\perp} \zeta_j}(\chi \eta_j) (\chi \eta_j, h) \overline{(\chi \eta_j, g)} d\nu(\chi) = \\ &= \sum_{j=0}^{p^{sN}-1} \int_X \mathbf{1}_{(K_{-N}^+)^{\perp} \xi_j}(\chi) (\chi, h) \overline{(\chi, g)} (\eta_j, h) \overline{(\eta_j, g)} d\nu(\chi). \end{aligned}$$

Since

$$\begin{aligned} (\eta_j, h) &= (\mathbf{r}_0^{\bar{\alpha}_0} \mathbf{r}_1^{\bar{\alpha}_1} \dots \mathbf{r}_{M-1}^{\bar{\alpha}_{M-1}}, \mathbf{a}_{-1} g_{-1} \dot{+} \mathbf{a}_{-2} g_{-2} \dot{+} \dots \dot{+} \mathbf{a}_{-l} g_{-l}) = 1, \\ (\eta_j, g) &= (\mathbf{r}_0^{\bar{\alpha}_0} \mathbf{r}_1^{\bar{\alpha}_1} \dots \mathbf{r}_{M-1}^{\bar{\alpha}_{M-1}}, \mathbf{b}_{-1} g_{-1} \dot{+} \mathbf{b}_{-2} g_{-2} \dot{+} \dots \dot{+} \mathbf{b}_{-l} g_{-l}) = 1, \end{aligned}$$

then

$$\begin{aligned} \int_E (\chi, h) \overline{(\chi, g)} d\nu(\chi) &= \sum_{j=0}^{p^{sN}-1} \int_{(K_{-N}^+)^{\perp} \xi_j} (\chi, h) \overline{(\chi, g)} d\nu(\chi) = \\ &= \int_{(K_0^+)^{\perp}} (\chi, h) \overline{(\chi, g)} d\nu(\chi) = \delta_{h,g} \end{aligned}$$

. \square

Theorem 7.1 *Let $K = F^{(s)}$ be a local field with positive characteristic p , $E \subset (K_M^+)^{\perp}$ an (N, M) -elementary set. If $|\hat{\varphi}(\chi)| = \mathbf{1}_E(\chi)$ on X then the system of shifts $(\varphi(x \dot{-} h))_{h \in H_0}$ is an orthonormal system on K .*

Proof. Let $\tilde{H}_0 \subset H_0$ be a finite set. Using the Plancherel equation we have

$$\begin{aligned} \int_K \varphi(x \dot{-} g) \overline{\varphi(x \dot{-} g)} d\mu(x) &= \int_X |\hat{\varphi}(\chi)|^2 \overline{(\chi, g)} (\chi, h) d\nu(\chi) = \int_E (\chi, h) \overline{(\chi, g)} d\nu(\chi) = \\ &= \sum_{j=0}^{p^{sN}-1} \int_{(K_{-N}^+)^{\perp} \zeta_j} (\chi, h) \overline{(\chi, g)} d\nu(\chi). \end{aligned}$$

Transform the inner integral

$$\begin{aligned} \int_{(K_{-N}^+)^{\perp} \zeta_j} (\chi, h) \overline{(\chi, g)} d\nu(\chi) &= \int_X \mathbf{1}_{(K_{-N}^+)^{\perp} \zeta_j}(\chi) (\chi, h) \overline{(\chi, g)} d\nu(\chi) = \\ &= \int_X \mathbf{1}_{(K_{-N}^+)^{\perp} \zeta_j}(\chi \eta_j) (\chi \eta_j, h \dot{-} g) d\nu(\chi) = \int_X \mathbf{1}_{(K_{-N}^+)^{\perp} \xi_j}(\chi) (\chi \eta_j, h \dot{-} g) d\nu(\chi) = \\ &= \int_{(K_{-N}^+)^{\perp} \xi_j} (\chi \eta_j, h \dot{-} g) d\nu(\chi). \end{aligned}$$

Repeating the arguments of lemma 7.1 we obtain

$$\int_K \varphi(x \dot{-} h) \overline{\varphi(x \dot{-} g)} d\mu(x) = \delta_{h,g}. \quad \square$$

8 Trees and wavelets

Let $K = F^{(s)}$ be a local field of characteristic p . In this section we reduce the problem of construction of step refinable function on the field K to construction of some tree.

We will consider some special class of refinable functions $\varphi(\chi)$ for which $|\hat{\varphi}(\chi)|$ is a characteristic function of a set. Define this class.

Definition 8.1 *A mask $m_0(\chi)$ is called N -elementary ($N \in \mathbb{N}_0$) if $m_0(\chi)$ is constant on cosets $(K_{-N}^+)^{\perp}\chi$, its modulus $m_0(\chi)$ has two values only: 0 and 1, and $m_0((K_{-N}^+)^{\perp}) = 1$. The refinable function $\varphi(x)$ with Fourier transform*

$$\hat{\varphi}(\chi) = \prod_{n=0}^{\infty} m_0(\chi \mathcal{A}^{-n})$$

is called N -elementary too. N -elementary function φ is called (N, M) -elementary if $\hat{\varphi}(\chi) \in \mathfrak{D}_{-N}(K_M^{\perp})$. In this case we will call the Fourier transform $\hat{\varphi}(\chi)$ (N, M) -elementary, also.

Definition 8.2 *Let $\tilde{E} = \bigsqcup_{\bar{\alpha}_{-1}, \bar{\alpha}_0} (K_{-1}^+)^{\perp} \mathbf{r}_{-1}^{\bar{\alpha}_{-1}} \mathbf{r}_0^{\bar{\alpha}_0} \subset (K_1^+)^{\perp}$ be an $(1, 1)$ -elementary set. We say that the set \tilde{E}_X is a periodic extension of \tilde{E} if*

$$\tilde{E}_X = \bigcup_{l=1}^{\infty} \bigsqcup_{\bar{\alpha}_1, \dots, \bar{\alpha}_l \in GF(p^s)} \tilde{E} \mathbf{r}_1^{\bar{\alpha}_1} \mathbf{r}_2^{\bar{\alpha}_2} \dots \mathbf{r}_l^{\bar{\alpha}_l}.$$

We say that the set \tilde{E} generates an $(1, M)$ elementary set E , if $\bigcap_{n=0}^{\infty} \tilde{E}_X \mathcal{A}^n = E$.

Let us write the set $GF(p^s)$ in the form

$$\{\mathbf{0}, \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_q, \bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_{p^s-q-1}\} = V, \quad \mathbf{0} = \mathbf{u}_0,$$

where $1 \leq q \leq p^s - 1$. We will consider the set V as a set of vertices. By $T(\mathbf{0}, \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_q, \bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_{p^s-q-1}) = T(V)$ we will denote a rooted tree on the set of vertices V , where $\mathbf{0}$ is a root, $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_q$ are first level vertices, $\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_{p^s-q-1}$ are remaining vertices.

For example for $p = 3, s = 2, q = 2, \mathbf{u}_1 = (2, 1), \mathbf{u}_2 = (1, 0)$ we have trees

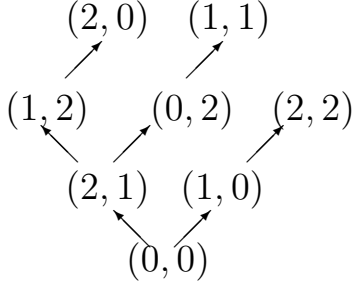


Figure 1

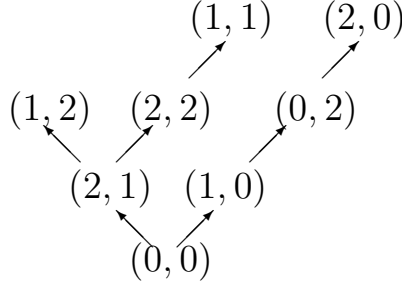


Figure 2

or

and so on.

For any tree path

$$P_j = (\mathbf{0} \rightarrow \mathbf{u}_j \rightarrow \bar{\alpha}_{l-1} \rightarrow \bar{\alpha}_{l-2} \rightarrow \cdots \rightarrow \bar{\alpha}_0 \rightarrow \bar{\alpha}_{-1})$$

we construct the set of cosets

$$(K_{-1}^+)^\perp \mathbf{r}_{-1}^{\mathbf{u}_j}, (K_{-1}^+)^\perp \mathbf{r}_{-1}^{\bar{\alpha}_{l-1}} \mathbf{r}_0^{\mathbf{u}_j}, (K_{-1}^+)^\perp \mathbf{r}_{-1}^{\bar{\alpha}_{l-2}} \mathbf{r}_0^{\bar{\alpha}_{l-1}}, \dots, (K_{-1}^+)^\perp \mathbf{r}_{-1}^{\bar{\alpha}_0} \mathbf{r}_0^{\bar{\alpha}_{-1}}, (K_{-1}^+)^\perp \mathbf{r}_{-1}^{\bar{\alpha}_{-1}} \mathbf{r}_0^{\bar{\alpha}_0}. \quad (8.1)$$

For example for the tree from Figure 2 and the path

$$(0, 0) \rightarrow (1, 0) \rightarrow (0, 2) \rightarrow (2, 0)$$

we have 3 cosets

$$(K_{-1}^+)^\perp \mathbf{r}_{-1}^{(1,0)}, (K_{-1}^+)^\perp \mathbf{r}_{-1}^{(0,2)} \mathbf{r}_0^{(1,0)}, (K_{-1}^+)^\perp \mathbf{r}_{-1}^{(2,0)} \mathbf{r}_0^{(0,2)},$$

for the path $(0, 0), (2, 1), (2, 2)$ we have two cosets

$$(K_{-1}^+)^\perp \mathbf{r}_{-1}^{(2,1)}, (K_{-1}^+)^\perp \mathbf{r}_{-1}^{(2,2)} \mathbf{r}_0^{(2,1)}.$$

We will represent the tree $T(V)$ as the tree $\begin{matrix} T_1 \cdots T_q \\ \swarrow \searrow \\ (0, 0) \end{matrix}$ where T_j are tree branches of $T(V)$ with \mathbf{u}_j as a root. By E_j denote a union of all cosets (8.1) for fixed j and set

$$\tilde{E} = \left(\bigsqcup_{j=1}^q E_j \right) \bigsqcup (K_{-1}^+)^\perp. \quad (8.2)$$

It is clear that \tilde{E} is an $(1, 1)$ elementary set and $\tilde{E} \subset (K_1^+)^\perp$.

Definition 8.3 Let \tilde{E}_X be a periodic extension of \tilde{E} . We say that the tree $T(V)$ generates a set E , if $E = \bigcap_{n=0}^{\infty} \tilde{E}_X \mathcal{A}^n$.

Lemma 8.1 *Let $T(V)$ be a rooted tree with $\mathbf{0} = (0, 0, \dots, 0)$ as a root. Let $E \subset X$ be a set generated by the tree $T(V)$, H a height of $T(V)$. Then E is an $(1, H - 2)$ -elementary set.*

Proof. Let us denote

$$m(\chi) = \mathbf{1}_{\tilde{E}_X}(\chi), \quad M(\chi) = \prod_{n=0}^{\infty} m(\chi \mathcal{A}^{-n}).$$

First we note that $M(\chi) = \mathbf{1}_E(\chi)$. Indeed

$$\mathbf{1}_E(\chi) = 1 \Leftrightarrow \chi \in E \Leftrightarrow \forall n, \chi \mathcal{A}^{-n} \in \tilde{E}_X \Leftrightarrow \forall n, \mathbf{1}_{\tilde{E}_X}(\chi \mathcal{A}^{-n}) = 1 \Leftrightarrow$$

$$\forall n, m(\chi \mathcal{A}^{-n}) = 1 \Leftrightarrow \prod_{n=0}^{\infty} m(\chi \mathcal{A}^{-n}) = 1 \Leftrightarrow M(\chi) = 1.$$

It means that $M(\chi) = \mathbf{1}_E(\chi)$. Now we will prove, that $\mathbf{1}_E(\chi) = 0$ for $\chi \in (K_{H-1}^+)^{\perp} \setminus (K_{H-2}^+)^{\perp}$. Since $\tilde{E}_X \supset (K_{-1}^+)^{\perp}$ it follows that $\mathbf{1}_{\tilde{E}_X}((K_{H-1}^+)^{\perp} \mathcal{A}^{-H}) = \mathbf{1}_{\tilde{E}_X}((K_{-1}^+)^{\perp}) = 1$. Consequently

$$\prod_{n=0}^{\infty} \mathbf{1}_{\tilde{E}_X}(\chi \mathcal{A}^{-n}) = \prod_{n=0}^{H-1} \mathbf{1}_{\tilde{E}_X}(\chi \mathcal{A}^{-n})$$

for $\chi \in (K_{H-1}^+)^{\perp} \setminus (K_{H-2}^+)^{\perp}$. Let us denote $m((K_{-1}^+)^{\perp} \mathbf{r}_{-1}^{\mathbf{i}} \mathbf{r}_0^{\mathbf{k}}) = \lambda_{\mathbf{i}, \mathbf{k}}$. By the definition of cosets (8.1) $m((K_{-1}^+)^{\perp} \mathbf{r}_{-1}^{\mathbf{i}} \mathbf{r}_0^{\mathbf{k}}) \neq 0 \Leftrightarrow$ the pair (\mathbf{k}, \mathbf{i}) is an edge of the tree $T(V)$.

We need prove that

$$\mathbf{1}_E((K_{-1}^+)^{\perp} \mathbf{r}_{-1}^{\bar{\alpha}_{-1}} \mathbf{r}_0^{\bar{\alpha}_0} \dots \mathbf{r}_{H-2}^{\bar{\alpha}_{H-2}}) = 0$$

for $\bar{\alpha}_{H-2} \neq 0$. Since \tilde{E}_X is a periodic extension of \tilde{E} it follows that the function $m(\chi) = \mathbf{1}_{\tilde{E}_X}(\chi)$ is periodic with any period $\mathbf{r}_1^{\bar{\alpha}_1} \mathbf{r}_2^{\bar{\alpha}_2} \dots \mathbf{r}_l^{\bar{\alpha}_l}$, $l \in \mathbb{N}$, i.e. $m(\chi \mathbf{r}_1^{\bar{\alpha}_1} \mathbf{r}_2^{\bar{\alpha}_2} \dots \mathbf{r}_l^{\bar{\alpha}_l}) = m(\chi)$ when $\chi \in (K_1^+)^{\perp}$. Using this fact we can write $M(\chi)$ for $\chi \in (K_{H-1}^+)^{\perp} \setminus (K_{H-2}^+)^{\perp}$ in the form

$$\begin{aligned} M((K_{-1}^+)^{\perp} \zeta) &= M((K_{-1}^+)^{\perp} \mathbf{r}_{-1}^{\bar{\alpha}_{-1}} \mathbf{r}_0^{\bar{\alpha}_0} \dots \mathbf{r}_{H-2}^{\bar{\alpha}_{H-2}}) = \\ &= m((K_{-1}^+)^{\perp} \mathbf{r}_{-1}^{\bar{\alpha}_{-1}} \mathbf{r}_0^{\bar{\alpha}_0}) m((K_{-1}^+)^{\perp} \mathbf{r}_{-1}^{\bar{\alpha}_0} \mathbf{r}_0^{\bar{\alpha}_1}) \dots \\ &\quad m((K_{-1}^+)^{\perp} \mathbf{r}_{-1}^{\bar{\alpha}_{H-3}} \mathbf{r}_0^{\bar{\alpha}_{H-2}}) m((K_{-1}^+)^{\perp} \mathbf{r}_{-1}^{\bar{\alpha}_{H-2}}) = \\ &= \lambda_{\bar{\alpha}_{-1}, \bar{\alpha}_0} \lambda_{\bar{\alpha}_0, \bar{\alpha}_1} \dots \lambda_{\bar{\alpha}_{H-3}, \bar{\alpha}_{H-2}} \lambda_{\bar{\alpha}_{H-2}, \mathbf{0}}, \quad \bar{\alpha}_{H-2} \neq \mathbf{0}. \end{aligned}$$

If $\lambda_{\bar{\alpha}_{H-2}, \mathbf{0}} = 0$ then $M((K_{-1}^+)^{\perp} \zeta) = 0$. Let $\lambda_{\bar{\alpha}_{H-2}, \mathbf{0}} \neq 0$. It means that $\bar{\alpha}_{H-2} = \mathbf{u}_j$ for some $j = \overline{1, q}$. If $\lambda_{\bar{\alpha}_{H-3}, \bar{\alpha}_{H-2}} = 0$ then $M((K_{-1}^+)^{\perp} \zeta) = 0$. Assume that $\lambda_{\bar{\alpha}_{H-3}, \bar{\alpha}_{H-2}} \neq 0$. It is true iff the pair $(\bar{\alpha}_{H-2}, \bar{\alpha}_{H-3})$ is an edge of $T(V)$. Repeating these arguments, we obtain a path

$$(\mathbf{0} \rightarrow \mathbf{u}_j = \bar{\alpha}_{H-2} \rightarrow \bar{\alpha}_{H-3} \rightarrow \cdots \rightarrow \bar{\alpha}_l)$$

of the tree $T(V)$. Since $\text{hight}(T) = H$ it follows that $l \geq 0$. Consequently $(\bar{\alpha}_l, \bar{\alpha}_{l-1})$ is not edge and $\lambda_{\bar{\alpha}_{l-1}, \bar{\alpha}_l} = 0$, where $l \geq 0$. It means that $M((K_{-1}^+)^{\perp} \zeta) = 0$.

Now we prove that E is $(1, H-2)$ -elementary set. Indeed, any path

$$(\mathbf{0} \rightarrow \mathbf{u}_j = \bar{\alpha}_{l-1} \rightarrow \bar{\alpha}_{l-2} \rightarrow \cdots \rightarrow \bar{\alpha}_0 \rightarrow \bar{\alpha}_{-1})$$

defines the coset $(K_{-1}^+)^{\perp} \mathbf{r}_{-1}^{\bar{\alpha}_{-1}} \mathbf{r}_0^{\bar{\alpha}_0} \cdots \mathbf{r}_{l-1}^{\bar{\alpha}_{l-1}} \subset E$. But for any $\bar{\alpha}_{-1} \in GF(p^s)$ there exists unique path with endpoint $\bar{\alpha}_{-1}$ and starting point zero. It means that E is $(1, H-2)$ -elementary set. \square

Theorem 8.1 *Let $M, s \in \mathbb{N}$, $p^s \geq 3$. Let $E \subset (K_M^+)^{\perp}$ be an $(1, M)$ -elementary set, $\hat{\varphi} \in \mathfrak{D}_{-1}((K_M^+)^{\perp})$, $|\hat{\varphi}(\chi)| = \mathbf{1}_E(\chi)$, $\hat{\varphi}(\chi)$ the solution of the equation*

$$\hat{\varphi}(\chi) = m_0(\chi) \hat{\varphi}(\chi \mathcal{A}^{-1}), \quad (8.3)$$

where $m_0(\chi)$ is an 1-elementary mask. Then there exists a rooted tree $T(V)$ with $\text{height}(T) = M+2$ that generates the set E .

Prof. Since the set E is $(1, M)$ -elementary set and $|\hat{\varphi}(\chi)| = \mathbf{1}_E(\chi)$, it follows from theorem 7.1 that the system $(\varphi(x \dot{-} h))_{h \in H_0}$ is an orthonormal system in $L_2(K)$. Using the theorem 6.1 we obtain that for any $\bar{\alpha}_{-1} \in GF(p^s)$

$$\sum_{\bar{\alpha}_0, \bar{\alpha}_1, \dots, \bar{\alpha}_{M-1} \in GF(p^s)} |\hat{\varphi}((K_{-1}^+)^{\perp} \mathbf{r}_{-1}^{\bar{\alpha}_{-1}} \mathbf{r}_0^{\bar{\alpha}_0} \cdots \mathbf{r}_{M-1}^{\bar{\alpha}_{M-1}})|^2 = 1. \quad (8.4)$$

Since $\hat{\varphi}$ is a solution of refinement equation (8.3) it follows from lemma 6.1 that for $\bar{\alpha}_{-1} \in GF(p^s)$

$$\sum_{\bar{\alpha}_0 \in GF(p^s)} |m_0((K_{-1}^+)^{\perp} \mathbf{r}_{-1}^{\bar{\alpha}_{-1}} \mathbf{r}_0^{\bar{\alpha}_0})|^2 = 1. \quad (8.5)$$

Let us denote $\lambda_{\bar{\alpha}_{-1}, \bar{\alpha}_0} := m_0((K_{-1}^+)^{\perp} \mathbf{r}_{-1}^{\bar{\alpha}_{-1}} \mathbf{r}_0^{\bar{\alpha}_0})$. Then we write (8.5) in the form

$$\sum_{\bar{\alpha}_0 \in GF(p^s)} |\lambda_{\bar{\alpha}_{-1}, \bar{\alpha}_0}|^2 = 1. \quad (8.6)$$

Since the mask $m_0(\chi)$ is 1-elementary it follows that $|\lambda_{\bar{\alpha}, \bar{\beta}}|$ take two values only: 0 or 1.

Now we will construct the tree T . Let \mathfrak{U} be a family of cosets $(K_{-1}^+)^\perp \zeta \subset (K_M^+)^\perp$ such that $\hat{\varphi}((K_{-1}^+)^\perp \zeta) \neq 0$ and $(K_{-1}^+)^\perp \notin \mathfrak{U}$. We can write a coset $(K_{-1}^+)^\perp \zeta \in \mathfrak{U}$ in the form

$$(K_{-1}^+)^\perp \mathbf{r}_{-1}^{\bar{\alpha}_{-1}} \mathbf{r}_0^{\bar{\alpha}_0} \dots \mathbf{r}_{M-1}^{\bar{\alpha}_{M-1}}.$$

If $(K_{-1}^+)^\perp \zeta \subset (K_l^+)^\perp \setminus (K_{l-1}^+)^\perp$ ($l \leq M$) then

$$(K_{-1}^+)^\perp \zeta = (K_{-1}^+)^\perp \mathbf{r}_{-1}^{\bar{\alpha}_{-1}} \mathbf{r}_0^{\bar{\alpha}_0} \dots \mathbf{r}_{l-1}^{\bar{\alpha}_{l-1}}, \quad \bar{\alpha}_{l-1} \neq \mathbf{0}.$$

Let $\mathbf{u} \neq \mathbf{0}$. By $T_{\mathbf{u}}$ we denote the set of vectors $(\mathbf{u}, \bar{\alpha}_{n-1}, \dots, \bar{\alpha}_0, \bar{\alpha}_{-1})$ for which $(K_{-1}^+)^\perp \mathbf{r}_{-1}^{\bar{\alpha}_{-1}} \mathbf{r}_0^{\bar{\alpha}_0} \dots \mathbf{r}_{n-1}^{\bar{\alpha}_{n-1}} \mathbf{r}_n^{\mathbf{u}} \in \mathfrak{U}$. We will name the vector $(\mathbf{u}, \bar{\alpha}_{n-1}, \dots, \bar{\alpha}_0, \bar{\alpha}_{-1})$ as a path too. So $T_{\mathbf{u}}$ is the set of pathes with starting point \mathbf{u} , for which $\hat{\varphi}((K_{-1}^+)^\perp \mathbf{r}_{-1}^{\bar{\alpha}_{-1}} \mathbf{r}_0^{\bar{\alpha}_0} \dots \mathbf{r}_{n-1}^{\bar{\alpha}_{n-1}} \mathbf{r}_n^{\mathbf{u}}) \neq 0$. Denote (it follow from (8.5)), if $\hat{\varphi}((K_{-1}^+)^\perp \mathbf{r}_{-1}^{\bar{\alpha}_{-1}} \mathbf{r}_0^{\bar{\alpha}_0} \dots \mathbf{r}_{n-1}^{\bar{\alpha}_{n-1}} \mathbf{r}_n^{\mathbf{u}}) \neq 0$ then $\hat{\varphi}((K_{-1}^+)^\perp \mathbf{r}_{-1}^{\bar{\alpha}_{-1}} \mathbf{r}_0^{\bar{\alpha}_0} \dots \mathbf{r}_{n-1}^{\bar{\alpha}_{n-1}} \mathbf{r}_n^{\mathbf{u}} \mathbf{r}_{n+1}^{\bar{\alpha}_{n+1}}) = 0$ for any $\bar{\alpha}_{n+1} \neq \mathbf{0}$. We will show that $T_{\mathbf{u}}$ is a rooted tree with \mathbf{u} as a root.

1) All vertices $\bar{\alpha}_j, \mathbf{u}$ of the path $(\mathbf{u}, \bar{\alpha}_{n-1}, \dots, \bar{\alpha}_0, \bar{\alpha}_{-1})$ are pairwise distinct. Indeed

$$\hat{\varphi}((K_{-1}^+)^\perp \mathbf{r}_{-1}^{\bar{\alpha}_{-1}} \mathbf{r}_0^{\bar{\alpha}_0} \dots \mathbf{r}_{n-1}^{\bar{\alpha}_{n-1}} \mathbf{r}_n^{\mathbf{u}}) = \lambda_{\bar{\alpha}_{-1}, \bar{\alpha}_0} \lambda_{\bar{\alpha}_0, \bar{\alpha}_1} \dots \lambda_{\bar{\alpha}_{n-1}, \mathbf{u}} \bar{\lambda}_{\mathbf{u}, \mathbf{0}} \neq 0, \quad \mathbf{u} \neq \mathbf{0}.$$

If $\bar{\alpha}_{n-1} = \mathbf{u}$ then $|\lambda_{\mathbf{u}, \mathbf{u}}| = |\lambda_{\mathbf{u}, \mathbf{0}}| = 1$ that contradicts the equation (8.5).

If $\bar{\alpha}_{n-1} = \mathbf{0}$ then $|\lambda_{\mathbf{0}, \mathbf{u}}| = |\lambda_{\mathbf{0}, \mathbf{0}}| = 1$ that contradicts the equation (8.5) too.

Consequently $\bar{\alpha}_{n-1} \notin \{\mathbf{0}, \mathbf{u}\}$. By analogy we obtain that

$$\bar{\alpha}_i \notin \{\mathbf{0}, \mathbf{u}, \bar{\alpha}_{n-1}, \dots, \bar{\alpha}_{i+2}, \bar{\alpha}_{i+1}\}.$$

2) If two pathes $(\mathbf{u}, \bar{\alpha}_{k-1}, \dots, \bar{\alpha}_0, \bar{\alpha}_{-1})$ and $(\mathbf{u}, \bar{\beta}_{l-1}, \dots, \bar{\beta}_0, \bar{\beta}_{-1})$ have the common subpath $(\mathbf{u}, \bar{\alpha}_{k-1}, \dots, \bar{\alpha}_{k-j+1}, \bar{\alpha}_{k-j}) = (\mathbf{u}, \bar{\beta}_{l-1}, \dots, \bar{\beta}_{l-j+1}, \bar{\beta}_{l-j})$ and $\bar{\alpha}_{k-j-1} \neq \bar{\beta}_{l-j-1}$ then $\{\bar{\alpha}_{-1}, \bar{\alpha}_0, \dots, \bar{\alpha}_{k-j-1}\} \cap \{\bar{\beta}_{-1}, \bar{\beta}_0, \dots, \bar{\beta}_{l-j-1}\} = \emptyset$. Indeed, let

$$\{\bar{\alpha}_{-1}, \bar{\alpha}_0, \dots, \bar{\alpha}_{k-j-1}\} \cap \{\bar{\beta}_{-1}, \bar{\beta}_0, \dots, \bar{\beta}_{l-j-1}\} \neq \emptyset.$$

Then there exists $\mathbf{v} \in \{\bar{\alpha}_{-1}, \bar{\alpha}_0, \dots, \bar{\alpha}_{k-j-1}\} \cap \{\bar{\beta}_{-1}, \bar{\beta}_0, \dots, \bar{\beta}_{l-j-1}\}$.

Assume that $\mathbf{v} \neq \bar{\alpha}_{k-j-1}$. Then $\mathbf{v} = \bar{\alpha}_\nu$, $-1 \leq \nu \leq k-j-2$ and $\mathbf{v} = \bar{\beta}_\mu$, $-1 \leq \mu \leq l-j-1$. It follows that

$$(\mathbf{u} = \bar{\alpha}_k, \dots, \bar{\alpha}_{k-j}, \bar{\alpha}_{k-j-1}, \dots, \bar{\alpha}_{\nu+1}, \bar{\alpha}_\nu = \bar{\beta}_\mu, \bar{\beta}_{\mu-1}, \dots, \bar{\beta}_0, \bar{\beta}_{-1}) \in T_{\mathbf{u}}$$

$$(\mathbf{u} = \bar{\beta}_l, \dots, \bar{\beta}_{l-j} = \bar{\alpha}_{k-j}, \bar{\beta}_{l-j-1}, \dots, \bar{\beta}_{\mu+1}, \bar{\beta}_\mu, \bar{\beta}_{\mu-1}, \dots, \bar{\beta}_0, \bar{\beta}_{-1}) \in T_{\mathbf{u}}.$$

So we have two different pathes with the same sheet $\bar{\beta}_{-1}$. But this contradicts theorem 6.1. This means that $T_{\mathbf{u}}$ has no cycles, consequently $T_{\mathbf{u}}$ is a graph with \mathbf{u} as a root.

3) By analogy we can proof that different trees $T_{\mathbf{u}}$ and $T_{\mathbf{v}}$ has no common vertices. It follows that the graph $T = (\mathbf{0}, T_{\mathbf{u}_1}, \dots, T_{\mathbf{u}_q})$ is a tree with $\mathbf{0}$ as a root.

4) It is evident that this tree generates refinable function $\hat{\varphi}$ with a mask m_0 . Show that $\text{height}(T) = M + 2$. Indeed, since $\hat{\varphi} \in \mathfrak{D}_{-1}((K_M^+)^{\perp})$ it follows that there exists a coset $(K_{-1}^+)^{\perp} r_{-1}^{\bar{\alpha}_{-1}} r_0^{\bar{\alpha}_0} \dots r_{M-1}^{\bar{\alpha}_{M-1}}$, $\bar{\alpha}_{M-1} \neq 0$ for which

$$|\hat{\varphi}((K_{-1}^+)^{\perp} r_{-1}^{\bar{\alpha}_{-1}} r_0^{\bar{\alpha}_0} \dots r_{M-1}^{\bar{\alpha}_{M-1}})| = 1.$$

This coset generates a path $(\mathbf{0}, \bar{\alpha}_{M-1} = \mathbf{u}, \bar{\alpha}_{M-2}, \dots, \bar{\alpha}_0, \bar{\alpha}_{-1})$ of T . This path contain $M+2$ vertex. It means that $\text{height}(T) \geq M+2$. On the other hand there isn't coset $(K_{-1}^+)^{\perp} \zeta \in \mathfrak{U}$ with condition $(K_{-1}^+)^{\perp} \zeta \subset (K_{M+1}^+)^{\perp} \setminus (K_M^+)^{\perp}$, consequently there isn't path with $L > M+2$. So $\text{height}(T) = M+2$. Since $\text{supp } \hat{\varphi}(\chi)$ is $(1, M)$ -elementary set, it follows that the set of all vertices of the tree T is the set $GF(p^s)$. The theorem is proved. \square

Definition 8.4 Let $T(V)$ be a rooted tree with $\mathbf{0}$ as a root, H a height of $T(V)$, $V = GF(p^s)$. Using cosets (8.1) we define the mask $m_0(\chi)$ in the subgroup $(K_1^+)^{\perp}$ as follows: $m_0((K_{-1}^+)^{\perp}) = 1$, $m_0((K_{-1}^+)^{\perp} \mathbf{r}_{-1}^{\mathbf{i}} \mathbf{r}_0^{\mathbf{j}}) = \lambda_{\mathbf{i}, \mathbf{j}}$, $|\lambda_{\mathbf{i}, \mathbf{j}}| = 1$ when $(K_{-1}^+)^{\perp} \mathbf{r}_{-1}^{\mathbf{i}} \mathbf{r}_0^{\mathbf{j}} \subset \tilde{E}$, (q.v. (8.2)), $|\lambda_{\mathbf{i}, \mathbf{j}}| = 0$ when $(K_{-1}^+)^{\perp} \mathbf{r}_{-1}^{\mathbf{i}} \mathbf{r}_0^{\mathbf{j}} \subset (K_1^+)^{\perp} \setminus \tilde{E}$. Let us extend the mask $m_0(\chi)$ on the $X \setminus (K_1^+)^{\perp}$ periodically, i.e. $m_0(\chi \mathbf{r}_1^{\bar{\alpha}_1} \mathbf{r}_2^{\bar{\alpha}_2} \dots \mathbf{r}_l^{\bar{\alpha}_l}) = m_0(\chi)$. Then we say that the tree $T(V)$ generates the mask $m_0(\chi)$. Set $\hat{\varphi}(\chi) = \prod_{n=0}^{\infty} m_0(\chi \mathcal{A}^{-n})$. It follows from Lemma 8.1 that

- 1) $\text{supp } \hat{\varphi}(\chi) \subset (K_{H-2}^+)^{\perp}$,
- 2) $\hat{\varphi}(\chi)$ is $(1, H-2)$ -elementary function,
- 3) $(\varphi(x \dot{-} h))_{h \in H_0}$ is an orthonormal system.

In this case we say that the tree $T(V)$ generates the refinable function $\varphi(x)$.

Theorem 8.2 Let $p \geq 2$ be a prime number, $s \in \mathbb{N}$, $p^s \geq 3$,

$$V = \{\mathbf{0}, \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_q, \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{p^s-q-1}\}$$

a set of vertices, $T(V)$ a rooted tree, $\mathbf{0}$ the root, $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_q$ a first level vertices. Let H be are height of $T(V)$. By $\varphi(x)$ denote the function generated by the $T(V)$. Then $\varphi(x)$ generate an orthogonal MRA on $F^{(s)}$.

Proof. Since $T(V)$ generates the function φ , it follows that 1) $\hat{\varphi} \in \mathfrak{D}_{-1}(K_1^+)^{\perp}$, 2) $\hat{\varphi}(\chi)$ is $(1, H-2)$ elementary function, 3) $\hat{\varphi}(\chi)$ is a solution of refinable equation (8.3), 4) $(\varphi(x-h))_{h \in H_0}$ is an orthonormal system. From the theorem 6.3 it follows that $\varphi(x)$ generates an orthogonal MRA. \square

Remark. Now we can give a simple algorithm for constructing non-Haar refinable function $\varphi(x)$. Let $T(V)$ be a tree on the set $V = GF(p^s)$. Construct a finite sequence $(\lambda_{\mathbf{i}, \mathbf{j}})_{\mathbf{i}, \mathbf{j} \in GF(p^s)}$ as follows: $\lambda_{\mathbf{0}, \mathbf{0}} = 1$, $|\lambda_{\mathbf{i}, \mathbf{j}}| = 1$ if the pair (\mathbf{j}, \mathbf{i}) is an edge of $T(f)$. For any vertex $\bar{\alpha}_{-1}$ we take the path $(\mathbf{0} = \bar{\alpha}_{l+1}, \mathbf{u}_j = \bar{\alpha}_l, \bar{\alpha}_{l-1}, \dots, \bar{\alpha}_0, \bar{\alpha}_{-1})$ and suppose

$$\hat{\varphi}((K_{-1}^+)^{\perp} \mathbf{r}_{-1}^{\bar{\alpha}_{-1}} \mathbf{r}_0^{\bar{\alpha}_0} \dots \mathbf{r}_{l-1}^{\bar{\alpha}_{l-1}} \mathbf{r}_l^{\bar{\alpha}_l} \mathbf{r}_{l+1}^{\mathbf{0}}) = \lambda_{\bar{\alpha}_{-1}, \bar{\alpha}_0} \cdot \lambda_{\bar{\alpha}_0, \bar{\alpha}_1} \cdot \dots \cdot \lambda_{\bar{\alpha}_{l-1}, \bar{\alpha}_l} \cdot \lambda_{\bar{\alpha}_l, \mathbf{0}}.$$

Otherwise we suppose $\hat{\varphi}((K_{-1}^+)^{\perp} \zeta) = 0$. Then φ generates an orthogonal MRA on the field $GF(p^s)$.

Example. Let $p = s = 2$. For these values we have trees

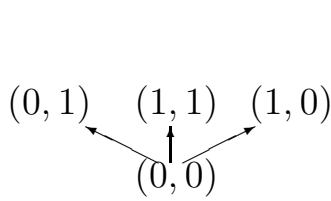


Figure 3

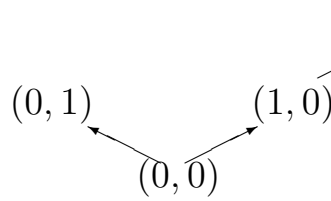


Figure 4

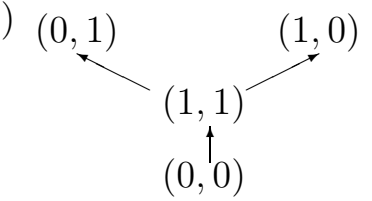


Figure 5

and so on. For the tree on figure 5 we obtain $\hat{\varphi}(\chi)$ in the form

$\hat{\varphi}(K_{-1}^+) = 1$, $\hat{\varphi}((K_{-1}^+)^{\perp} \mathbf{r}_{-1}^{(1,1)}) = \lambda_{1,1}$, $\hat{\varphi}((K_{-1}^+)^{\perp} \mathbf{r}_{-1}^{(0,1)} \mathbf{r}_0^{(1,1)}) = \lambda_{0,1}$, $\hat{\varphi}((K_{-1}^+)^{\perp} \mathbf{r}_{-1}^{(1,0)} \mathbf{r}_0^{(1,1)}) = \lambda_{1,0}$. $|\lambda_{i,j}| = 1$ and $\hat{\varphi}((K_{-1}^+)^{\perp} \zeta) = 0$ otherwise. Suppose for simplicity $\lambda_{i,j} = 1$. Then we can calculate the scaling function

$$\begin{aligned} \varphi(x) &= \int_X \hat{\varphi}(\chi)(\chi, x) d\nu(\chi) = \int_{(K_{-1}^+)^{\perp}} (\chi, x) d\nu(\chi) + \int_{(K_{-1}^+)^{\perp} \mathbf{r}_{-1}^{(1,1)}} (\chi, x) d\nu(\chi) + \\ &+ \int_{(K_{-1}^+)^{\perp} \mathbf{r}_{-1}^{(0,1)} \mathbf{r}_0^{(1,1)}} (\chi, x) d\nu(\chi) + \int_{(K_{-1}^+)^{\perp} \mathbf{r}_{-1}^{(1,0)} \mathbf{r}_0^{(1,1)}} (\chi, x) d\nu(\chi) = 2^{-2}(\mathbf{1}_{K_{-1}^+}(x) + \\ &+ \mathbf{r}_{-1}^{(1,1)}(x) \mathbf{1}_{K_{-1}^+}(x) + \mathbf{r}_{-1}^{(0,1)}(x) \mathbf{r}_0^{(1,1)}(x) \mathbf{1}_{K_{-1}^+}(x) + \mathbf{r}_{-1}^{(1,0)}(x) \mathbf{r}_0^{(1,1)}(x) \mathbf{1}_{K_{-1}^+}(x)) = \mathbf{1}_E(x) \end{aligned}$$

where

$$\begin{aligned} E &= K_1^+ \bigsqcup (K_1^+ \dot{+} (0,0)g_{-1}) \bigsqcup (K_1^+ \dot{+} (1,1)g_{-1}) \bigsqcup (K_1^+ \dot{+} (1,0)g_{-1} \dot{+} (1,1)g_0) \\ &\bigsqcup (K_1^+ \dot{+} (0,1)g_{-1} \dot{+} (1,1)g_0). \end{aligned}$$

We can consider additive group K^+ as product $\mathfrak{G} \times \mathfrak{G}$ of Cantor groups. In this case $\hat{\varphi}$ and φ may be defined on the product $\mathfrak{G}_1^\perp \times \mathfrak{G}_1^\perp$ and $\mathfrak{G}_{-1} \times \mathfrak{G}_{-1}$ respectively by the tables

0	0	1	0
0	0	0	1
0	1	0	0
1	0	0	0

$\mathfrak{G}_0^\perp \quad \mathfrak{G}_1^\perp$ \mathfrak{G}_{-1}^\perp

Figure 6. $\hat{\varphi}$

0	0	1	0
0	0	0	1
0	1	0	0
1	0	0	0

$\mathfrak{G}_0 \quad \mathfrak{G}_{-1}$ \mathfrak{G}_1

Figure 7. φ

Since $\text{supp} \hat{\varphi} \neq (K_0^+)^\perp$ and $\text{supp} \varphi \neq (K_0^+)$, it follows that φ generates non-Haar MRA. From this example we see that MRA on local field gives an effective method to construct multidimensional step wavelets.

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